EUCLID'S PLANE GEOMETRY,

BOOKS III—VI.,

PRACTICALLY APPLIED;

OR

GRADATIONS IN EUCLID, PART II.,

WITH

ALGEBRAICAL AND ARITHMETICAL ILLUSTRATIONS, EXPLANATORY NOTES, AND A SYNOPTICAL INDEX TO THE SIX BOOKS, SHEWING THE USES OF THE PROPOSITIONS, &c.,

BY HENRY GREEN, A.M.

"THERE IS (GENTLE READER) NOTHING (THE WORD OF GOD ONELY SET APART) WHICH SO MUCH BEAUTEFIETH AND ADORNETH THE SOULE AND MINDE OF MAN, AS DOTH THE KNOWLEDGE OF GOOD ARTES AND SCIENCES." *Billingsley's Euclid*, A. D. 1570.

MANCHESTER: JOHN HEYWOOD, 143, DEANSGATE.
LONDON: SIMPKIN, MARSHALL, & CO.

1861.

TO WILLIAM FAIRBAIRN, C.E, LL.D., F.R.S, F.G.S.,

CORRESPONDING MEMBER OF THE NATIONAL INSTITUTE OF
FRANCE, AND OF THE ROYAL ACADEMY OF TURIN,
CHEVALIER OF THE LEGION OF HONOUR,
ETC., ETC.;

WHOSE AIMS IN LIFE
HAVE LED HIM TO ENTERPRISES OF PUBLIC UTILITY,
AND WHOSE WRITINGS AND EXAMPLE
SHOW THE PRACTICAL USES OF SCIENTIFIC KNOWLEDGE;

THIS WORK,
BY HIS PERMISSION, IS INSCRIBED;
AND WITH EVERY SENTIMENT OF RESPECT.

PREFACE.

The Preface to Bks. I and II of the "Gradations in Euclid" contains most of the observations which can be required from the Author. One of them he now repeats;—"At the present day nearly every edition of EUCLID's Elements must be more or less a compilation, in which the Author draws freely on the labours of his predecessors. 'The Gradations' are, in a great degree, of this character; and an open acknowledgment will suffice, once for all, to repel any charge of intentionally claiming what belongs to others. It is affectation to pretend to great originality on a subject which has, like Geometry, for so many centuries exercised men's minds."

Originality has not been the Author's aim, but usefulness. With this purpose before him, he has endeavoured to show how few, if any, Geometrical Truths are destitute of a practical application. The stigma which some have attempted to fasten on what they name, "mere Mathematical Theories," is thus removed, and the Science, instead of being repulsive from its dry abstractions, is invested with the ever-abiding charm of being both the foundation and the builder-up of very many most important practical results. In fact, there is scarcely a branch of human knowledge, from the art of sketching an outline, to that of spanning and measuring the heavens themselves, which does not depend for its vigour and comprehensiveness on the aids which Geometry furnishes.

Step by step man ascends the Himalayas and compasses the earth; step by step is the Mathematician's course. As in the First Part, so in this Second Part of the "Gradations," the same method has been pursued. Prefixed to the Proof are references to the principles that have to be employed, and often quotations of the very words in which Euclid embodies those principles. These are not so fully given, indeed, as in the First Part, for it is presumed that familiarity with leading principles and propositions has been already attained. Occasionally too, in the steps of the Construction and Demonstration, the special reference is not made in the margin to the evidence on which an argument or a conclusion rests; but the Learner will scarcely find this any obstacle, if he has mastered what he has read.

It will be well for the Learner thoroughly to consider the references, before he proceeds to the Particular Enunciation, the Construction and the Demonstration of the Proposition;—indeed, were he of himself to put together the truths with which he is supplied, and to see how the new truth is to be deduced from them, he would derive the best assistance, that from the reasoning of his own mind, to understand and appreciate the fuller proof of the formal demonstration. The Memory, no doubt, is a most valuable power in acquiring any kind of knowledge, but in Mathematics especially it is the understanding and the reasoning faculty that are employed to most advantage and developed with greater exactness.

As an instance of the method recommended, let the Learner take that important Proposition, 35, III, "*If two st. lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.*" After weighing these words of the General Enunciation, let him call to

mind the several truths contained in the references under the heads, "Con.," construction, and "Dem.," demonstration; and if he has forgotten any of them let him turn back to the very propositions numbered, as 10, I., 3, III., 47, I., &c., and carefully think them over. He thus burnishes up his old weapons; and now let him try to trace out the connexion between the propositions referred to, and to ascertain how they lead to the new proposition which he seeks to establish. He will say to himself, here are several undoubted truths and facts presented to me;—I have already accepted them as principles of Geometrical Reasoning,—and they are now given that I may demonstrate some other truth, or solve some other problem. Can I not, with the implements provided, build this new hou se and see how, like the others, it is composed of indestructible materials?

He may rely, that by thus exercising his judgment, he will do more, than any mere effort of memory can do, for really understanding and retaining mathematical truths.

A very full Table of signs and abbreviations is given, and this should be consulted until they have become well known.

Some of the Demonstrations, as in 8, V, and 15, V, have been shortened. By the time the Learner has mastered so much of the Geometry, he will readily perceive the connexion of the argument, and not require the entire fulness of which it is capable; or, if he should, he may be expected to supply it from his own resources.

The Index was a subject of some consideration. An alphabetical Index had been prepared, but it was rejected, because it would have occupied too much space. The advantage of learners appeared to be more promoted by having the whole of the General Enunciations

of Euclid's Geometry brought together under the heads of Problems and Theorems, with their respective illustrations, applications, and uses. An alphabetical Index would have facilitated references to particular truths, but the consecutive or synoptical Index conduces more to the understanding of the whole work, and to the tracing out of the connexions of its parts.

A word or two to those who, from inexperience, do not understand the difficulty of avoiding errors of the press in a work where many signs, abbreviations, and references are used. As a most justly celebrated mathematician* has observed,—The Table of Corrigenda, at the end of the volume, "may convey an impression that the work is incorrectly printed, which is not the case;" and he adds, "If every mathematical work, at its completion, had the fruits of some years of examination presented to the reader, I know of none which would not have lists as large in proportion to their size and the number of symbols contained in them as the present one." On this subject the author will simply remark, that should "the Euclid Practically applied" attain a second edition, these and some other faults will be carefully amended.

In his Mathematical Preface, "written at his poor House at Mortlake, Anno 1570, February 9," "John Dee, of London," addressed himself "to the vnfained Lovers of Truthe and constant Studentes of Noble Sciences;" "he hartely wisheth them grace from heaven and most prosperous successe in all their honest attemptes and exercises." So, with him, I say to all who value good learning, "I commit you vnto God's Mercyfull direction for the rest; hartily beseechyng hym, to prosper your Studyes and honest Intentes to His Glory and the Commodity of our Country."

October 1, 1861.

* De Morgan, in his *Differential and Integral Calculus*, A.D. 1842.

SYMBOLICAL NOTATION AND ABBREVIATIONS.

I.—*Signs common to Arithmetic, Algebra, and Geometry.*

$\because$ because.	$+$ *plus*, add, together with.
$\therefore$ therefore.	$-$ *minus*, subtract, take away.
$\therefore$ wherefore.	$\sim$ difference between.
$=$ equals, *or* equal.	$\times$ *into*, multiply.
$\neq$ not equal to, or unequal.	$\div$ *by*, divide.
$>$ greater than.	$\surd$ root.
$\ngtr$ not greater than.	$:$ ratio.
$<$ less than.	$: = :$ equality of ratios.
$\nless$ not less than.	$: : : :$ proportion.

$: \ : \ :$ Numbers or Quantities in Progression

The signs $>$, $\ngtr$, $<$, $\nless$, between ratios, as A : B $>$ C : D, or A : B $\ngtr$ C : D, or A : B $<$ C : D, or A : B $\nless$ C : D, denote that the one ratio *is greater than*, or *not greater than*, *less than*, or *not less than*, the other ratio, according to the sign.

II.—*Geometrical Signs.*

$\cdot$ a point. *	$\triangle$ triangle.
$\mid$ straight line.	$\square$ parallelogram.
$\parallel$ parallel, parallel to.	$\square$ square, $\square$ rectangle.
$\angle$ angle.	$\odot$ circle.
$\perp$ perpendicular to, at rt. $\angle$s.	$\odot$ce circumference.

* When an *s* is added to a sign, or to an abbreviation, the plural is denoted

A *single* capital letter, as A, or B, denotes the point A, or the point B; but sometimes, as in Bks. V and VI, the quantity, or magnitude, A, B, C, &c.

Two capital letters, as A B, or C D, denote the straight line A B, or C D; but when the letters indicate opposite angles, they denote a parallelogram, or a rectangle, or a polygon, as the figure will show.

A capital letter, or two capital letters, with the *numeral* ² just above to the right hand, as A², or A B², denote not the square *of* A, A B, but the square *on* A or A B.

Capital letters, with a point between them, as AB.CD, denote, not the product of AB multiplied by CD, but the rectangle formed by two of its sides meeting in a common point.

III.—*Additional Algebraic Expressions.*

M	Magnitude.	n or p	another multiple.
m	multiple.	$m+n$	the sum of the quantities m & n.
m A &c.	multiple of A &c.		
m A, m B, &c.	equimultiples of A,B &c.	mn	the product of $m \times n$.
		mn A	a multiple of A by mn
m (A + B)	multiple of (A+B)	$(m+n)$ A	a multiple of A by $m+n$.
m (A — B)	multiple of (A—B).		
m (A+B—C)	multiple of the excess of (A+B) above C.	*pt.*	part.
		sub-m	submultiple.

III.—*Abbreviations.*

Add......*Addendo,* by adding.
App......Application of a Prop.
Appl.........*Applicando,* by applying
C. or Con..Construction.
C. 1 &c...Step 1 &c. of the Construction.
Conc......Conclusion, inference.
Cor.......Corollary.
Dat.......*Datum,* or *data.*
D. or Dem. Demonstration.
D. 1 &c...Step 1 &c. of the Dem.
E. or Exp..Exposition, or Particular Enunciation of a Prop.
Ex.......Example.
Gen......General Enunciation.
H. or Hyp.Hypothesis of a Prop.
H. 1 &c...Step 1 &c. of the Hyp.
L ....Line.
M. or Mag. Magnitude.
P. or Prop. Proposition.
Pon.........*Ponendo,* by placing, by position.
Prob......Problem.
Proced ...*Precedendo,* by going on.
Prel......Preliminary.
Prod.........Product.
Pst. or Psts.Postulate, or Postulates
Quæs.....*Quæsitum* or *Quæsita.*
R............Ratio.
Rec.......... Recapitulation.
Remk... .Remark to be made.
Sch.......*Scholium* or *Scholia.*
Sim.......So, similarly, by similar reasoning.
S. or Sol...Solution of a Problem.
Sum. ....*Sume,* take away.
Sup.......Suppose, *or* Let.
Super.....*Superponendo,* by superdosition.
Theor.....Theorem.

Q. E. D. *quod erat demonstrandum*, which was the thing to be proved.

Q. E. F. *quod erat faciendum*, which was the thing to be done.

adj. . . . . . .adjacent.
ad imposs. .*ad impossibile*, to an impossibility.
a fort. . . . .*a fortiori*, by a stronger reason.
alt. . . . . .altitude.
altr. . . . . . .alternate.
antec . . . .antecedent.
ang ....angular.
assum. . . . .*assumendo*, by adopting.
bis. . . . . . . .bisects, *or* bisect.
bisd. . . . . . .bisected.
bisg. . . . . . .bisecting.
cen. . . . . . .centre.
ch. . . . . . . .chord.
com. . . . . . .common.
comp- . . . .compound.
compl. . . . .complement.
con. sup. . .contrary supposition.
c. scr. . . . . .circumscribe.
c. scg. . . . .circumscribing.
conseq. . ....consequent.
cont. . . . . .continued..
conterm ...conterminous.
contn. ......contain, or contained.
descr. . . . .describe, or described.
descg. . . . .describing.
diag. . . . . .diagonal.
diam. . . . . .diameter.
diff. . . . . . .difference.
dist. . . . . . .distance, or distant.
div. . . . . . .divide, or divided.
dupl. . . . .duplicate.
eq. . . . . . . .equal or equally.
eq. ang. . . .equiangular.
eq. lat. . . . .equilateral.
ex. ab. . . . .*ex absurdo*, by an absurdity.
ex gr, ...*exempli gratiâ*, for example's sake.
ext. . . . . . .exterior, or exteriorly.
extn. . . . . .externally.
extr . . . . . .extremity, or extremities.
fig. . . . . . .figure.
gr. . . . . . . .greater.
homol. . .homologous.
hyp. . . . . .hypotenuse.
incl. . . . . . .include, included.
indef. . . . .indefinitely.
inscr. . . . .inscribe, inscribed.
int. . . . . . .interior.
inters. . . . .intersect, intersection.
intr ....internal, internally.
m or mult. . . } multiple
magn. . . . .magnitude.
meas. . . . .measure.
mid. . . . . . .middle.
ob. . . . . . . .obtuse.
opp. . . . . . .opposite.
par. . . . . . .parallel.
parlm. . . . .parallelogram.
pent . . . . . .pentagon.
perp. . . . . .perpendicular.

pos. . . . . . .position.
prod. . . . .produce, produced.
propl. . . . .proportional.
pt. . . . . . . .part.
qu. ang. . .quadrangular.
qu. lat. . . .quadrilateral.
rad. . . . . . .radius.
rat. . . . . . .ratio.
recip. . . . .reciprocal.
rect. . . . . . .rectangle.
rectl. . . . . .rectilineal.
rectr. . . . . .rectangular.
reg. . . . . . .regular.
rem. . . . . . .remaining.
resp. . . . . .respective.
rt. . . . . . . .right.
sect. . . . . .sector.
seg. . . . . . .segment.
sem. c. . .semicircle.
sem. cirf. . .semicircumference.
sim. . . . . . .similar to, similarly.
sim. sit. . . .similarly situated.
sq. . . . . . . .square.
st. . . . . . . .straight.
suppl. . . . . .supplemental.
tang. . . . . .tangent.
rap. . . . . .trapezium.
undiv. . . . .undivided.
uneq. . . . . .unequal, or unequally.
vert. . . . . . .vertex, vertical.

GRADATIONS IN EUCLID.

BOOK III.

TREATING OF THOSE PROPERTIES OF THE CIRCLE, AND OF STRAIGHT LINES IN AND ABOUT IT, WHICH CAN BE DEDUCED FROM THE FIRST AND SECOND BOOKS.

A circle, strictly speaking, signifies the space bounded by a circumference, but in this book the term is employed sometimes to denote that space, and at other times, the circumference itself.

Euclid, too, occasionally assumes from experimental knowledge, certain properties of the circle, which a more rigid and exact method of reasoning would have established before using them. This is the case in the first Proposition itself, where it is taken for granted that the perpendicular to the chord of an arc will meet the circle in two points. In some instances also the method of *indirect* demonstration is adopted, when the more satisfactory method of *direct* proof is available; examples of this occur in Props. 2, 13, 16 and 36.

By restricting the meaning of the term angle to an opening formed by two conterminous lines, and less than two right angles,

Euclid renders some of his demonstrations, as that of Prop. 21, more cumbersome than they need be.

The Properties of the right-angled triangle, of the circle, and of certain lines in and about a circle, as the radius, the sine, the tangent and the secant, have laid the foundations of by far the most extensive branch of Mathematics. Trigonometry, Plane and Spherical, resting on these properties and at first "confined to the solution of one general problem, has now spread its uses over the whole of the immense domains of the mathematical and physical sciences."—LARDNER'S Trigonometry, p. 3.

The Learner may therefore enter on the study of this Third Book with the assurance, that he is about to cross the threshold of one of the most important parts of Plane Geometry. "The influence, indeed, of the properties of the circle upon abstract mathematical analysis has been so great that an attempt to describe the manner in which the means of expression derived from this figure has been used, would fill a volume."

The quaint English *Editio Princeps* of Euclid, published in 1570, thus opens to the Reader the Summary of Bk. III.

"This third booke of Euclide entreateth of the most perfect figure, which is a circle. Wherefore it is much more to be estemed then the two bookes goyng before, in which he did set forth the most simple proprieties of rightlined figures. For sciences take their dignities of the worthynes of the matter that they entreat of. But of al figures the circle is of most absolute perfection, whose proprieties and passions are here set forth, and most certainly demôstrated. Here also is entreated of right lines subtended tò arkes in circles: also of angles set both at the circumference and at the centre of a circle, and of the varietie and difference of them. Wherfore the readyng of this booke, is very profitable to the attayning to the knowledge of chordes and arkes. It teacheth moreover which are circles contingêt, and

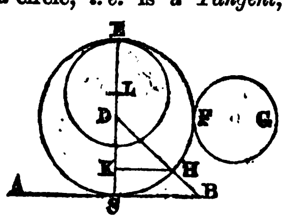
which are cutting the one the other: and also that the angle of contingence is the least of all acute rightlined angles: and that the diameter in a circle is the longest line that can be drawen in a circle. Farther in it may we learne how, three pointes beyng geuen how soever (so that they be not set in a right line) may be drawen a circle passing by them all three. Agayne, how in a solide body, as in a Sphere, Cube, or such lyke, may be found the two opposite pointes. Whiche is a thyng very necessary and commodious, chiefly for those that shall make instrumentes seruyng to Astronomy and other artes."—BILLINGSLEY'S EUCLID, fol. 81.

DEFINITIONS.

1. *Equal circles* are those of which the diameters are equal, *or*, from the centres of which the straight lines to the circumferences are equal.

The *criterion* of the equality of circles is that their diameters or their radii are equal;—but this is neither a Definition nor an Axiom; properly it is a Theorem, the truth of which may be proved by *superposition*;—for if centre be placed on centre and the equal radii or diameters on each other, the circumference of the one will in each point coincide with the circumference of the other,—and thus the space included by one circumference will equal the space included by the other.

2. A straight line is said to touch a circle, *i.e.* is a *Tangent*, when it meets the circle, and being produced does not cut it; as AB tangent to EFC in C.



The point in which the straight line meets the circle is the *point of contact;* the straight line "does not cut," *i. e.* does not pass into the circle. A *Secant* is a straight line which, when it meets the circle and is produced, passes into the circle, *i.e.,* cuts or crosses the circumference; as BHD, secant to EFC in H.

The terms Tangent and Secant are often restricted in meaning;—the *first* to the line which by one extremity touches an extremity of the diameter at right angles to it, and which has its other extremity terminated by a straight line, the Secant, from the centre of the circle across the circumference; the *second* to the line from the centre, across the circumference, and terminated by the tangent. Thus BC is the tangent and DB the secant of the arc CH, or of the angle CDB measured by the arc; the name Cosine being given to the space DK cut off between the centre D and the sine; and Versed Sine to the space KC between the sine and the tangent point C.

The term *Sine* denotes a perpendicular to a diameter from the point where the secant crosses the circumference; as HK.

The terms Tangent, Secant, Sine, &c., thus restricted, were of continual use in Trigonometry; and with a widely extended meaning are now constantly employed. The process is instructive, by which extension has been given to Trigonometrical Symbols, and may thus be briefly stated;

1. Sine, cosine, &c., at first denoted lines so named drawn in and about a circle, with reference to an angle at the centre, and measured by its arc, each angle having a different sine, &c., according as the radius of the circle was increased or diminished in length.

2. To avoid these continual diversities, *that* radius was supposed always to be a unit, or rather the unit of measurement for the other lines; and the secant, sines, &c., to be multiples or fractional parts of that unit; thus the sine of 60°, being equal to the radius, was unity, or 1, and the sine of 30° was ½, or ·5. The names sines, cosines, &c., in this way lost their first meaning; they denoted, not lines, but the numerical ratios of those lines to the radius, and were abstract numbers.

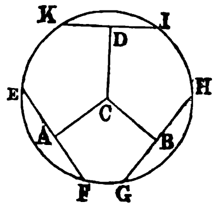
3. Another step was to represent the angle itself by an abstract number. Degrees and minutes had been the measure of the central angle, *that* angle was measured by its arc, and the arc bore a numerical ratio to the unit of measurement, the radius.

4. A fourth step made the process perfect. Hitherto the sum of the angles could not exceed four right angles, but this limit also was to be passed. The idea of a line revolving round a point, and continuing its rotation after a revolution had been completed, originated the method of using angles consisting of more than four right angles.

Thus angles, sines, &c., were all represented by numbers; and though the old names were retained, Trigonometry which at first was a simple application of Geometrical truths, and which still rests on Geometry for its foundation, became a branch of the higher Arithmetic, and has its operations conducted on arithmetical and algebraical principles.

3. Circles are said to touch one another, which meet, but do not cut one another. Thus the circle of which L is the centre, touches EFC in E, and circle G touches it in F.

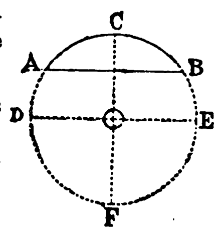
4. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal; thus E F and G H are equally distant from C, when perp. C A = perp. C B.



5. And the straight line on which the greater perpendicular falls, is said to be farther from the centre; thus I K is farther from C than G H is, because the perp. C D > perp. C B.

As the distance from the vertex of a triangle to its base is measured by a perpendicular, so the distance of a straight line from the centre of a circle is the perpendicular drawn to it from the centre. A Proposition analagous to Props. 7 and 8, Book III., would explain "why the perpendicular from a point on a straight line is called the distance from that line."

6. A *segment* of a circle is the figure contained by a straight line, and the circumference which it cuts off; as the fig. A B C A.

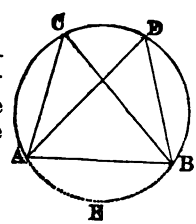


"A figure included by an arc and its chord is a segment."—LARDNER.

The straight line of a segment, as A B, is named the *chord*, and the circumference it cuts off the *arc*, as A C B. Every chord, except a diameter, divides the circle into unequal portions. The division of the circle by a diameter makes *semi-circles*, D C E, and D F E; by two diameters, bisecting at right angles, *quadrants*, as D F, F E. Two arcs A C B, A F B, having the same chord A B, evidently make up the whole circle.

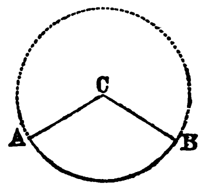
7. The *angle of a segment* is that which is contained by the straight line and the circumference; as angle A B C contained by A B and the arc B C A.

8. An *angle in a segment* is the angle contained by two straight lines drawn from any point of the circumference of the segment, to the extremities of the straight line which is the base of the segment; as ∠ A C B, or ∠ A D B.



9. An *angle* is said to *insist*, or, *stand upon* the circumference intercepted between the straight lines that contain the angle; as $\angle$ ACB on arc AEB.

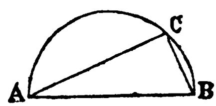
10. A *sector* of a circle is the figure contained by two straight lines drawn from the centre, and by the circumference between them; as ACB.

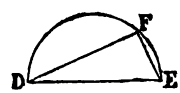


Sectors are equal when they have equal radii and equal angles; for by superposition their boundaries coincide in every respect.

Certain Sectors of a definite size are known by definite names; as the *Quadrant*, a sector of which the arc is 90°; the *Sextant*, of 60°; the *Octant*, of 45°.

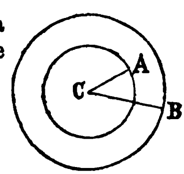
11. Similar segments of circles are those in which the angles are equal, or which contain equal angles; as $\angle$ A C B = $\angle$ D F E.





The idea of similarity here introduced belongs to all regular figures: all squares, equilateral triangles, hexagons, &c. are similar, though not equal,—the similarity depending on the equality of the angles. Figures become *identical* when their sides as well as their angles are respectively equal each to each.

12. Concentric circles are such as have a common centre; thus, circle A and circle B have the same centre C.



AXIOM A.

"If the distance of a point from the centre of a circle be less than the radius of the circle, the point is within the circle; and if the distance of a point from the centre of a circle is greater than the radius, the point is without the circle."—HOSE, p. 300. See also SCH. 2, PR. 1, III.

PROPOSITIONS.

Prop. 1.—Prob.

To find the centre of a given circle.

Sol.—Pst. 1. Let it be granted that a st. line may be drawn from any one point to any other point.

10, I. To bisect a given finite st. line.

11, I. To draw a st. line at rt. $\angle$s to a given st. line from a given point in the same.

Pst. 2. A terminated line may be produced to any length in a st. line.

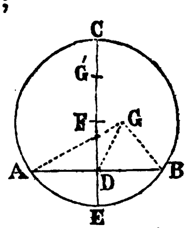
Dem.—Def. 15, I. A $\odot$ is a plane figure contained by one line, which is called the circumference, and is such that all st. lines drawn from a certain point within the figure to the $\odot$ce are equal to one another.

8, I. If two $\triangle$s have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle contained by the two sides equal to them of the other.

Def. 10, I. When a st. line standing on another st. line makes the adj. $\angle$s equal to each other, each of these $\angle$s is called a rt. $\angle$; and the st. line which stands on the other is called a perpendicular to it.

Ax. 1. Things which are equal to the same thing are equal to one another.

E. 1	Dat.	Let ABC be the given $\odot$;
2	Quæs.	to find its centre.
C. 1	Pst. 1. 10, I.	Draw any chord AB, and bis. it in D;
2	11. I.	at D draw DC $\perp$ AB;
3	Pst. 2. 10, I.	prod. CD to E, and bis. CE in F;
4	Sol.	then · F is the cen. of $\odot$ ABC.
5	*Assum.*	If not, take G as the cen;
6	Pst. 1.	and join GA, GD, GB.



D. 1	C.1, Def.15, I	In $\triangle$s ADG, BDG $\because$ DA = DB, GA = GB & GD com.
2	8, I. Def.10, I	$\therefore$ $\angle$ ADG = $\angle$ BDG, $\therefore$ BDG is a rt. $\angle$.
3	C 2. Ax. 1.	But $\because$ $\angle$ FDB is a rt. $\angle$ $\therefore$ $\angle$FDB = $\angle$ BDG, *i. e.* the less = the gr.;
4	*ad imposs.*	an impossibility $\therefore$ G is not the cen.
5	*Sim.*	So, no point out of CE is the cen.
6	C. 3,	And $\because$ CE is bis. in F,
7	Def. 15, I.	$\therefore$ any other point in CE is not the cen.,
8	Conc.	$\therefore$ No point but F is cen. of $\odot$ ABC.

Q. E. F.

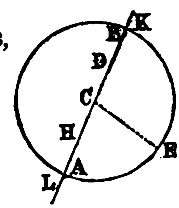
Cor.—*If in a* $\odot$ *a st. line,* CE, *bisects another,* AB, *at rt.* $\angle$*s, the cen. of the* $\odot$ *is in the line,* CE, *which bisects the other.*

Sch.—1. When the $\cdot$ G′ is taken in the diam. CE, the demonstration holds good only if G′ coincides with F; should this not be the case, it is evident that G′C $\neq$ G′E $\therefore$ G′ is not the cen.

2. The rigour of the reasoning would have been greatly promoted, if Euclid, previously to the above Problem, had established the following proposition; *Any point,* D, *fig.* 1, F, *fig.* 2. *being assumed within a* $\odot$, *a rt. line,* HD *or* HF, *drawn through it and produced indefinitely in both directions, will meet the* $\odot$ *in two points, and not in more; and every point of the line between these two points of intersection will be within the* $\odot$, *and every point beyond them without it.*—Lardner's Euclid. p. 91

I. Let HD through D, also pass through the cen. C.

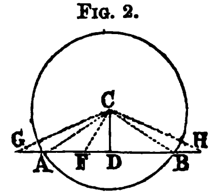
Fig. 1.



C 1	Pst. 3.	Prod. HD indef. in both directions;
& 2	Cor. 3, I.	& from C make CK, CL each $>$ CB,
D		CA;
3	Ax. A. III.	$\therefore$ the $\cdot$ s K & L are *without* the $\odot$.
4	3. I.	Also from C make CD, CH each $<$ CB, CA;
5	Ax. A. III.	$\therefore$ the $\cdot$ s H & D are *within* the $\odot$.
6	3. I.	Lastly, from C make CB, CA each = rad. CE;
7	Def. 15, I.	$\therefore$ the $\cdot$ s A & B *on* the $\odot$ce.

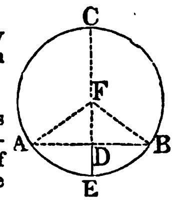
II. Let HF through F *not* pass through the cen. C.

Fig. 2.



C 1	11 I.	From C draw CD $\perp$ FH;
& 2	19, I. *a fort*	$\because$ CD $<$ CF & CF $<$
D		rad. CA $\therefore$ CD $<$ CA.
3	Cor.3. 47, I.	Find AD^2 = rad.^{2} – CD^2, *i. e.* CA^2 – CD^2,
4	3, I. Pst. 1.	from D set DA = DB & join CA, CB.
5	D.3.&47, I.	$\because$ BD^2 or DA^2 + CD^2 = rad.^{2} $\therefore$ CA = CB = rad.

6	Def. 15, I.	∴ the · s A & B each *on* the ⊙ce.
7	C. & Ax. A	Again ∵ D F < D A < rad. C A ∴ F is *within* the ⊙.
8	19, I.	And ∵ from G or H in AB produced, CG, or CH > rad. C A ;
9	Ax. A.	∴ the · s G & H are *without* the ⊙.
10	Conc.	Hence the st. line H F meets the ⊙ce only in two points.
11	Recap.	∴ *Any point being assumed, &c.* Q. E. D.



USE.—1. Practically the centre of a ⊙ is found, by bisecting any chord, A B with a perp. C E, terminated in the ⊙ce ; and C E being bisd. in F, F is the centre.

2. The First Prop. bk. III. is applicable to all cases in which the centre of any circular object, as of the horizontal section of a tree, may be required. A circular disk of metal, a wheel, a flower-bed, any object possessing the circular form will have its centre found in the same way.

PROP. 2.—THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

CON. 1, III. Pst. 1 & 2.

DEM. Def. 15, I.

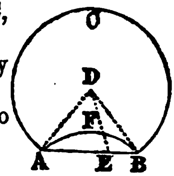
5, I. The angles at the base of an isosc. △ are equal to each other; and if the equal sides be produced, the ∠s on the other side of the base shall be equal.

16, I. If one side of a △ be produced the ext. ∠ is greater than either of the int. opp. angles.

19, 1. The gr. ∠ of every △ is subtended by the gr. side, or has the gr. side opposite to it.

E 1	Hyp.	Let A B C be ⊙, and A B any two · s in the ⊙ce;
2	Conc.	the st. line from A to B *within* the ⊙.

SUP.—If not *within*, and it is possible, let it be *without*, as A E B.



C. 1	1. III. Pst. 1.	Find D cen. of ⊙ ABC, & join DB, DA ;
2	*Assum.*	in ⊙ce A F B take any · F;
3	Pst. 1 & 2.	join D F, and prod. it to meet A B in · E.
D. 1	Def. 15, I. 5, I.	Then ∵ D A = D B, ∴ ∠ DAB = ∠ DBA;
2	C. & 16. I.	& ∵ A E in △ D A E is prod. to B; ∴ ext. ∠ DEB > int. & opp. ∠ DAE.
3	D. 1.	But ∠ DAE = ∠ DBE; ∴ ∠ DEB > ∠ DBE;
4	19. I.	and ∴ DB > DE.
5	Def. 15, I. *ad imp.*	Now DB = DF ∴ D F > DE; an impossibility;
6	Conc.	∴ the line from A to B *not without* the ⊙.
7	*Sim.*	So, AB does *not* fall *upon* the ⊙ce;
8	Conc.	∴ AB is *within* the ⊙.
9	Recap.	∵ *If any two points be taken &c.*

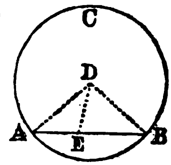
Q. E. D.

COR. 1.—*A st. line,* A B, *cannot cut the ⊙ce of a ⊙ in more points than two;* for, every st. line joining any two points in the ⊙ce falls within the ⊙, neither co-inciding with any other points in the ⊙ce, nor meeting it except in the two given points.

COR. 2.—*A st. line which touches a circle meets it only in one point.*

COR. 3.—A circle is concave towards its centre.

SCH.—Instead of the *ex absurdo* demonstration of this Proposition, a *direct* method of proof, founded on Axiom A, bk. III. was given by COMMANDINE, who lived between A.D. 1509 and 1575 ; he applied himself to mathematics at Verona, and in 1572, at Pesaro, published Euclid's Elements in fifteen books, in Latin.



C. 1	*Assum.*	In the given line A B, take any · E;
2	1, III. Pst. I.	find D cen. of the ⊙; and join D A, D E, D B.
D. 1	Def. 15, I. 5, I	∵ in △ D A B, D A = D B; ∴ ∠ D A B = ∠ D B A;
2	C.	and ∵ in △ A E D, A E is prod. to B,
3	10, I.	∴ ext. ∠ D E B > int. and opp. ∠ D A E;
4	D. 1.	but ∠ D A E = ∠ D B E; ∴ ∠ D E B > ∠ D B E.
5	D. 4. 19, I.	But ∠ D E B > ∠ D B E ∴ D B > D E;
6	Remk.	*i. e.* D E the dist. of E from D < D B the rad.
7	Ax. A. III.	∴ the · E is *within* the circle.
8	*Sim.*	So is every · between A and B;
9	Conc.	∴ the line A B, joining A and B falls *within* the ⊙.

Q. E. D.

USE.—On this proposition are grounded those which show, that a circle touches a st. line in only one point; for if the st. line touched two points of the ⊙ce the st. line would be drawn from one point of the ⊙ce to the other, and consequently would fall within the circle, contrary to the very definition of such a line, that it does not cut the circumference. THEODOSIUS of Tripolis, a mathematician who lived some time after the reign of Trajan, compiled a work on the Properties of the Sphere and of the circles described on its surface, an edition of which was published at Oxford in 1675: he used Prop. 2. bk. III. to demonstrate that a Globe resting on a plane surface cannot touch the plane in any but a single point; otherwise the plane would enter the globe.

PROP. 3.—THEOR.

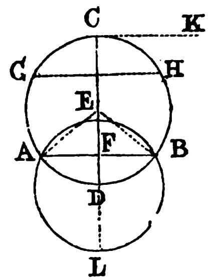
If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cuts it at right angles, it shall bisect it.

CON. 1, III. Pst. 1.

DEM. Def. 15, I. 8, I. Def. 10, I. 5. I.

26, I. If two △s have two ∠s of the one equal to two ∠s of the other, each to each, and one side equal to one side, viz., either the sides adj. to the equal ∠s in each, or the sides opp. to them, then shall the other sides be equal, each to each, and also the third ∠ of the one to the third ∠ of the other.

E. 1	Hyp. 1.	Let ABC be a circle;
2	„ 2	& CD a straight line through E the cen. bisg. AB a st. line not through E;
3	Conc. 1.	then CD cuts AB ⊥ in F;
4	„ 2.	& if ⊥ in F, AF = FB.
C. 1	1, III. Pst. 1.	Take E cen. of ⨀, & join EA, EB.



I. *The st. line* CD *shall cut* AB *at rt.* ∠*s.*

D. 1	H 2. & C.	∵ AF = FB, and FE com. to △s AFE, BFE;
2	Def. 15, I. 8, I.	& ∵ EA=EB ∴ ∠AFE=∠BFE;
3	Remk. Def. 10, I.	& being adj. ∠s ∴ ∠s AFE & BFE rt. ∠s.
4	Conc.	∴ CD through E, bisg. AB not through E, cuts AB at rt. ∠s.

II. *Also* CD *cutting* AB *at rt.* ∠*s, bis.* AB, *i.e.* AF=FB.

D. 1	Def. 15, I. 5, I.	∵ EA = EB ∴ ∠ EAF = ∠ EBF;
2	Def. 10, I.	and rt. ∠ AFE = rt. ∠ BFE;
3	D. 1, 2 & C.	Thus ∠s EAF, AFE = ∠s EBF, BFE, & EF com.;
4	26, I.	∴ AF = FB, *i.e.* AB is bisd.
5	Rec.	∴ *If a st. line be drawn, &c.* Q.E.D.

Cor. 1.—*A st. line* EF, *bisecting any chord,* AB, *at rt.* ∠*s, passes through the centre* E *of the circle.*

Cor. 2.—*All chords,* AB, GH *par. to the tang.* CK, *at either extremity of the diam.* CD, *are bisected by the diam.*

Cor. 3.—*The st. line which bisects* AB, *the com. chord of two circles,* ACB, ALB, *at rt.* ∠*s, passes through* E *&* D, *the centres of both circles.*

Cor. 4.—*When in a circle there are several chords*, AB, GH *par. to each other the* locus *of their points of bisection is in that diam.* CD, *which is at rt.* $\angle$*s to them; and if the line*, CD, *which bisects one chord be a perp. that line bisects all the par. chords at rt.* $\angle$*s.*

Sch.—This Prop. might be more briefly given; "If a diam. cut any other chord at rt. $\angle$s, it shall bisect it; and conversely, if a diam. bisect any other chord, it shall cut it at rt. $\angle$s."

Use and App.—The use of this Prop. extends to the various cases, in which arcs, or circles, have to be drawn, and their centres ascertained; as,—

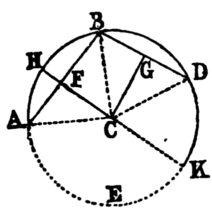
I. *Given a circle*, A B C, *to find its centre*, as in the last figure.

S.	1	Pst. 1. 10, I. 11, I.	Draw any ch. A B; bis. it in F by the perp. E F;
	2	Pst. 2.	prod. E F to $\cdot$s C, D, forming diam. C D;
	3	10. I.	bis. C D in E; and E is the centre.
D.	1	C. 1. Cor. 3, III.	$\because$ C D bis. A B $\perp$ $\therefore$ D C through the centre;
	2	C. 2., Def. 15, I.	$\because$ C D is a diam. $\therefore$ its mid. $\cdot$ E cen. of $\odot$

Q. E. F.

II. *Given an arc* A B D, *to find the cen. of the* $\odot$ *of which it is an arc.*

S.	1	*Assum.*	Take any $\cdot$ B, between the extremities of the arc;
	2	Pst. 1.	from B draw chords to A & D;
	3	10, I. & 11, I.	bis. AB & BD by perps. from F & G, meeting in C;
	4	Sol.	then, the intersecting $\cdot$ C is the centre.
D.	1	Cor. 3, III.	$\because$ the cen. lies in F C & also in G C;
	2	Conc.	$\therefore$ it must be in the intersecting $\cdot$ C.

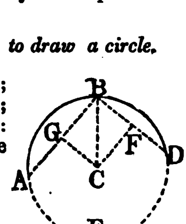


Q. E. F.

N.B. With rad. C A, or C D, the circle, A B D E, may be completed.

III. *Through three points*, A, B, D, *not in a st. line, to draw a circle.*

S.	1	Pst. 1. 10, I.	Join A B, B D, and bis. them in F, G;
	2	11. I.	at F & G raise perps. intersecting in G;
	3	Cor. 3, III.	C is the centre of $\odot$ through A, B, D:
	4	Pst. 3.	and with rad. A C, or B C the circle may be drawn.



N.B. The Dem. is given in Prop. 9, III.

IV. In Trigonometry we show 1st, by this Prop., in the last figure but one, that the half chord F B or F A of an arc A H B is perp. to the semidiam. C H, and consequently is the *Sine* of the half arc H B, or H A; 2nd, that the sides of a triangle (4, VI.) have the same ratio as the sines of their opp. angles.

V. In the last figure but one, That part of the perp. to the chord which passes through the centre and is intercepted between the centre and the chord, namely, CF, is called the *versed sine* (see note to Def. 2, p. 4); and the radius, semichord, and versed sine form respectively the hypotenuse, base and perp. of a rt. angled triangle, and by 47, I., when any two are measured, or given, the third may be found;—for,

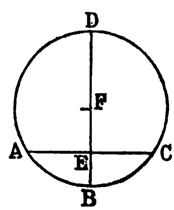
$$rad. = \sqrt{semich.^2 + vers.\ sine^2}; \quad semich. = \sqrt{rad.^2 - vers.\ sine^2};$$
and $vers.\ sine = \sqrt{rad.^2 - semich.^2}$

Prop. 4.—Theor.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Con.—10, I. 1, III. Pst. 1.

Dem.—Def. 15, I. 3, III.
Ax. 11. All rt. ∠s are equal.
Ax. 9. The whole is greater than its part.

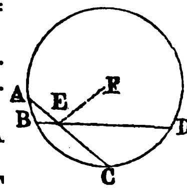


E.1	Hyp.1.	Let A B C be a circle;
2	„ 2.	& A C, B D two st. lines cutting in E, but not both through cen. F;
3	Conc.	then AC, B D do not bis. one another; *i.e.* E not mid. • both of A C & B D.

Sup. I. *Let* B D *pass through the centre and* A C *not.*

C.	10, I. Def. 15, I.	Bis. B D in F, then F cen. of ⊙.
D.	Def. 15, I.	∵ F B = F D ∴ E not mid. • of B D; *i. e.* B D not bisected in E by A C.

SUP. II. *Let neither* AC *nor* BD *pass through the centre.*



E.1	Hyp.	If possible let both AE = EC and BE = ED.
C.	1, III. Pst. 1.	Find F the cen. and join FE.
D.1	H.	∵ FE through cen. F, bis. AC not through F,
2	3. III.	∴ FE cuts AC ⊥, and FEA is a rt. ∠.
3	H.	Again, ∵ FE through cen. F bis. BD not through F,
4	3, III.	∴ FE cuts BD ⊥, and FEB is a rt. ∠.
5	Ax. 11.	∴ FEA a rt. ∠ = FEB a rt. ∠,
6	Remk. Ax. 9.	*i. e.* a part = the whole;—an impossibility;
7	Conc.	∴ AC and BD do not bisect each other.
8	Rec.	∴ *If in a circle two st. lines, &c.* Q. E. D.

COR.—*No parallelogram except a rectangle can be inscribed in a circle.*

D.1	C. & 34. I.	∵ the diags. are diams. ∴ the diags. bis. each other in their centres.
2	Def. 15, I.	∴ the diags. are equal.
3	8, I. 34, I.	And ∵ the suppl. ∠s are equal;
4	Conc.	∴ each ∠ is a rt. ∠, and all the ∠s are rt. ∠s. *i. e.* the inscribed fig. must be a rectangle.

USE.—The fourth Prop. has been employed to determine the eccentricity of the Sun's apparent path, or of the Earth's orbit described in a year.

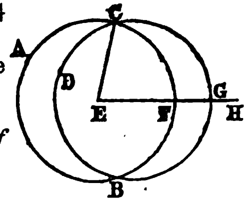
In an eccentric wheel the distance of the fixed point, or centre of rotation, E, round which the revolution is performed, from F, the centre of the wheel, will be found in the same way.

PROP. 5.—THEOR.

If two circles cut one another, they shall not have the same centre.

CON.—Pst. 1.

DEM.—Def. 15, I. Axs. 1 & 9.

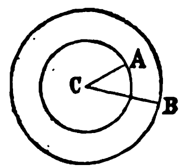


E.1	Hyp.	Let the two ⊙s ABC, CDG cut in ·s B & C;
2	Conc	then, they have not the same centre.

SUP.—*If possible let* E *be the cen. of both circles.*

C.1	Pst. 1.	Join EC, & from E draw a st. line EFGH, meeting the ⊙s in ·s F & G.
D.1	Sup. Def. 15, I.	∵ E cen. of ⊙ ABC ∴ EC = EF.
2	" "	∵ E cen. of ⊙ CDG ∴ EC = EG.
3	D. & Ax. 1.	But EC = EF ∴ EF = EG,
4	Remk. Ax. 9.	*i.e.* the less = gr.; an impossibility;
5	Conc.	∴ E not the com. cen. of ⊙s ABC, CDG.
6	Rec.	∴ *If circles cut one another, &c.*

Q. E. D.



SCH.—"This proposition may be better announced thus: 'Concentric circles cannot meet, and that which has the lesser radius will be included within the other.'"—LARDNER, p. 94.

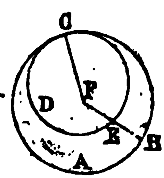
For ∵ CA < CB ∴ ⊙ A *within* ⊙ B.

PROP. 6.—THEOR.

If one circle touch another internally, they shall not have the same centre.

CON.—Pst. 1.

DEM.—Def. 15, I. Axs. 1 & 9.



E.1	Hyp.	Let ⊙ CDE touch ⊙ ABC internally in C;
2	Conc.	they have not the same cen.

Sup.—*If they have the same centre let it be* • F.

C.	Pst. 1.	Join FG, and from F draw a st. line FEB, meeting the $\odot$s in •s E & B.
D.1	Sup. Def. 15, I.	$\because$ F cen. of $\odot$ ABC $\therefore$ FC = FB;
2	” ”	$\because$ F cen. of $\odot$ CDE $\therefore$ FC = FE;
3	D. 1. & Ax. 1.	But FC = FB $\therefore$ FE = FB,
4	Remk. Ax. 9.	*i.e*, the less = gr.; which is impossible;
5	Conc.	$\therefore$ F not the com .cen. of $\odot$s ABC, CDE.
6	Rec.	$\therefore$ *If one circle touch another internally, &c.* Q.E.D.

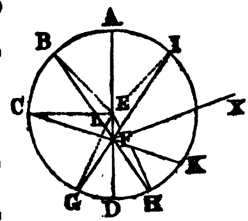
Sch.—Props. 5 & 6 may be combined into one; "circles with a common centre do not touch either externally or internally:" for the circle with the less radius will have every point within the circumference of the other, and consequently does not meet the other in any point whatever.

Prop. 7.—Theor.

If any point which is not the centre be taken in the diameter of a circle, then, 1st, *of all the straight lines which can be drawn from it to the circumference,* the greatest *is that in which the centre is, and the other part of that diameter is* the least; *and,* 2nd, *of any other st. lines, that which is nearer to the line which passes through the centre is always* greater *than the one more remote; also* 3rd, *those lines which make equal angles with the diameter are* equal; *and,* 4th, *from the same point there can be drawn* only two equal *st. lines, one upon each side of the diameter.*

Con.—Pst. 1. 23, I. At a given • in a given line, to make a rectl. ∠ equal to a given rectl. ∠.
3, I. From the gr. of two given lines to cut off a part equal to the less.

Dem.—20. I. Any two sides of a △ are together gr. than the third side.
Def. 15, I. Ax. 9.
24, I. If two △s have two sides of the one equal to the two sides of the other, each to each, but the ∠ contained by the two sides of the one gr. than the ∠ contained by the two sides of the other, the base of that which has the gr. ∠ shall be gr. than the base of the other.
Ax. 5. If equals be taken from unequals the remainders are unequal.
4. I. If two △s have each two sides and the included ∠ of the one equal to two sides and the included ∠ of the other, the △s are equal in every respect.—Ax. 1.



E.1	Hyp. 1.	Let ABCD be a ⊙ and AD its diam.
2	„ 2.	and in diam. AD any • F not the cen.
3	„ 3.	and let E be the centre;
		First, then of all st. lines FB, FC, FD &c. from F to ⊙ce.
4	Conc. 1.	FA through cen. E the greatest,
5	„ 2.	FD the other pt. of diam. AD the least;
		Second, of any other lines,
6	„ 3.	FB nearer to FA > FC more remote and FC > FG;
		Third, lines FB, FI making equal ∠s with diam. AD.
7	„ 4.	the line FB = the line FI.
		and *Fourth*, of lines from the same • F to the ⊙ce,
8	„ 5.	only two eq. lines, FG, FH, one on each side of diam. AD.
C.1	Pst. 1.	Join BE, CE, GE.

I.—*A line* FA *through cen.* E > *any other line as* BF.

D.1	20. I.	∵ two sides of a △ > third side ∴ BE + EF > BF,
2	Def.15, I.	but AE = BE ∴ AE + EF *i.e.* FA > FB.

II.—*The other part of the diam.*, FD < *any other line* FG.

D.1	Def. 15, I. C.	Again ∵ BE = CE & FE com. to △s BEF, CEF,
2	Ax. 9. 24, I.	but ∠BEF > ∠CEF ∴ BF > CF;
3	*Sim.*	So CF > GF and GF > DF.
4	20. I, Def. 15, I	Also ∵ GF + FE > GE, and EG = ED;
5		∴ GF + FE > ED.
6	*Sub.* Ax. 5.	take away com. pt. FE ∴ rem. GF > rem. FD.
7	Conc.	∴ Of all st. lines from F a • not the cen. to the ⊙ce,
		FA through cen. E is the greatest, and FD the least; and

III.—*Line* BF *nearer diam.* > CF *more remote* & CF > GF.

IV.—*The lines* FB, FI, *making with* AD ∠ BFA = ∠IFA, *are equal.*

Suppose one to be gr. *i. e.* FB > FI.

C.	3, I. Pst. 1.	Make FL = FI and join EL and EI.
D.1	C. & H.	In △s FLE, FIE ∵ FE com., FL = FI, & ∠ BFA = ∠ IFA,
2	4, I.	∴ EL = EI;
3	Def.15,I.Ax.I.	but EI = EC ∴ EC = EL.
	Ax. 9.	*i. e.* a part = the whole, which is absurd.
4	Conc.	∴ Neither FB nor FI the gr.; *i. e.* FB = FI.

V.—*Also only two equal lines*, FG, FH, *from* · F *to the* ⊙ce.

C.	23. I. Pst. 1.	At E in EF make ∠ FEH = ∠ FEG, and join FH.
D.1	Def. 15, I. C.	∵ GE = HE, EF com. and ∠ GEF = ∠ HEF,
2	4, I.	∴ FG = FH,
3	Remk.	And from F to ⊙ce. no other line = FG,
4	*Sup.*	If possible let FK = FG;
5	C. Ax. 1.	∵ FK = FG = FH ∴ FK = FH,
6	Remk.	*i. e.* a line nearer the diam. = a line more remote,
7	Case III.	which has been proved to be impossible.
8	Rec.	Therefore, *If any point which is not the centre, &c.* Q. E. D.

SCH.—If from a point, within a circle, not the centre, as F, a st. line of indef. length, as FX, revolve so as in each part of its revolution to be terminated or cut off by the circumference, as in A, B, C, G, D; its *maximum* length FA, will be attained when it coincides with that part of the diam. AD, in which E the centre is; and its *minimum* FD, when it coincides with the other part of the diam.; and the nearer FX, is to the *maximum* the greater it is, as FB, and the nearer to the *minimum*, the less, as FG.

USE.—THEODOSIUS, mentioned p. 11, by aid of this proposition proves that, if from the pole of the world, which is not the pole of the horizon, (for the zenith is *its* pole) several arcs of great circles be drawn to the circumference of the horizon, the *greatest* arc shall be that part of the meridian which passes through the zenith.

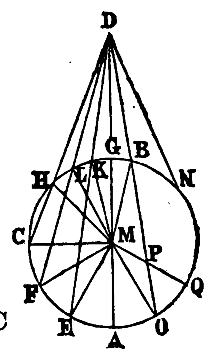
By this proposition we may also prove that the Earth being in Aphelion is at the greatest distance from the Sun, in Perihelion, at the least; and so for all the other planets.

PROP. 8.—THEOR.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; 1st, those which make equal angles with the line passing through the centre are equal; *2nd, of those which fall upon the concave circumference,* the greatest *is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre is always* greater *than one more remote; but, 3rd, of those which fall upon the convex circumference,* the least *is that between the given point without the circle and the diameter, and of the rest, that which is nearer to the least is always* less *than one more remote; and, 4th, only* two equal lines *can be drawn from the same point to the circumference, one upon each side of the line which passes through the centre.*

CON. 1, III, Pst. 1. 3, I. 23, I.

DEM. 4, I. Def. 15, I. Ax. 1 & 2. 20, I. Ax. 9. 24, I. Ax. 5.
21. I. If from the ends of the side of a $\triangle$ there be drawn two st. lines to a · within the $\triangle$, these lines shall be less than the other two sides of the $\triangle$, but shall contain a gr. $\angle$.



E.1	Hyp. 1.	Let ABC be a $\odot$ and D any · without it;
2	„ 2.	from · D let st. lines DA, DE, DF, DC, and DO be drawn to the $\odot$ce;
3	„ 3.	Of these let DA pass through cen. M,
4	„ 4.	and DE, DO make with DA, $\angle$ ADE $=$ $\angle$ ADO;
5	Conc. 1.	the line DE $=$ line DO;
		1st. *Of lines incident on* AEFC *the concave* $\odot$*ce,*
6	„ 2.	the greatest is DA passing through cen. M;
7	„ 3.	and any line nearer DA is $>$ a line more remote, *i. e.* DE $>$ DF and DF $>$ DC:

2nd. *But of lines incident on* HLKG *the convex* $\odot$*ce*,

E.8	Conc. 4.	the least is DG, between • D and diam. GA;
9	„ 5.	and any line nearer DG $<$ a line more remote, *i. e.* DK $<$ DL and DL $<$ DH.
10	„ 6.	Also only two equal lines, DB and DK, from • D to $\odot$ce, can be drawn, one upon each side of DA.
C.1	1, III.	Take M the cen. of $\odot$ ABC
2	Pst. 1.	and join MO, ME, MF, MC, MH, ML, and MG.

I. *The st. lines* DE *and* DO *are equal, making equal* $\angle$s *with* DA.

	Sup.	If possible let DO $>$ DE.
C.	3, I. Pst. 1.	Make DP $=$ DE, and join ME, MP.
D.1	C. and H.	In $\triangle$s MPD, MED $\because$ MD com. DP $=$ DE and $\angle$ ADE $=$ $\angle$ ADO or ADP;
2	4, I.	$\therefore$ MP $=$ ME.
3	Def.15, I. Ax.1	But MQ $=$ ME $\therefore$ MP $=$ MQ,
4	Ax. 9.	*i. e.* a part $=$ the whole, which is absurd;
5	Conc.	$\therefore$ Neither DO nor DE gr., *i. e.* DO $=$ DE.

II. *Of lines incident on* AEFC *the concave* $\odot$*ce*, DA *through the cen. is greatest, and* DE, *nearer* DA *is* $>$ DF *more remote, and* DF $>$ DC:

D.1	Def. 15, I. *Add.*	$\because$ AM $=$ ME, to each add MD;
2	Ax. 2.	$\therefore$ AM $+$ MD, *i.e.* AD $=$ EM $+$ MD;
3	20, I.	but EM $+$ MD $>$ ED $\therefore$ AD $>$ ED.
4	Def.15, I. Ax.9	Again $\because$ ME $=$ MF, MD com. & $\angle$ EMD $>$ $\angle$ FMD,
5	24, I.	$\therefore$ ED $>$ FD.
6	*Sim.*	In like manner FD $>$ CD.
7	Conc.	$\therefore$ DA through cen. M is the greatest, DE $>$ DF & DF $>$ DC.

III. *Of lines incident on* HLKG *the convex* $\odot$*ce*, DG, *between* D *and the diam.* AG, *is the least*, DK $<$ DL & DL $<$ DH.

D.1	20, I. Def. 15, I.	$\because$ MK $+$ KD $>$ MD, & MK $=$ MG;
2	Ax. 5.	$\therefore$ rem. KD $>$ rem. GD; *i.e.* GD $<$ KD,
3	C.	& $\because$ MLD is a $\triangle$, and from • s M, D, extrs. of one side MD, MK & DK are drawn to • K within the $\triangle$;

D.4	21, I.	$\therefore$ MK + KD < ML + LD;
5	Def. 15, I. Ax. 5	but MK = ML $\therefore$ rem. KD < rem. LD.
6	*Sim.*	In like manner DL < DH:
7	Conc.	$\therefore$ DG the least, DK < DL & DL < DH.

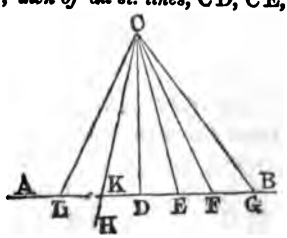
IV. *Also only two equal st. lines can be drawn from the* • D *to the* ⊙*ce, one on each side of* AD *the line which passes through cen.* M.

C.	23, I. Pst. 1.	At M in MD make ∠ DMB = ∠ DMK, & join DB
D.1	C.	$\because$ in △s KMD, BMD, MK = MB, MD com. & ∠ KMD = ∠ BMD;
2	4, I.	$\therefore$ DK = DB.
3	Remk.	But besides DB no other line from • D to the ⊙ce = DK;
4	Sup.	If there can, let it be DN;
5	D. 4 & 3, Ax. 1.	$\because$ DK = DN & also = DB $\therefore$ DB = DN
6	Remk.	*i.e.* a line nearer the least = one more remote;
7	D. II. 7.	but this has been proved to be impossible;
8	Conc.	$\therefore$ No line but DB = DK.
9	Rec.	Therefore, *If any point be taken*, &c. Q. E. D.

Sch. 1.—The *concave* and *convex* parts of a circumference are determined by the tangents from a point external to the circumference. All the parts of the circumference which are farther from the external point than the tangent points of the circle, are *concave* with respect to that point; and those nearer to the external point than the tangent points are *convex*.

Sch. 2.—A proposition, analagous to Prop. 7 & 8, explains why in Def. 4, III. and in 14 & 15, III. the perp. from a point out of a st. line to the line is called the *distance* of the point from or to the line.—Hose, p. 302.

If any • C *be taken out of a st. line* AB; *then of all st. lines,* CD, CE, CF, CG, *from that* • *to the st. line,* the least, CD, *shall be that which is perp. to the st. line; and of the rest,* CE, CF, CG, *that which is nearer, as* CE, *to the perp., shall always be* less *than that,* CF *or* CG, *which is more remote; and from this* • C *there can be drawn to the st. line* AB, *only one st. line = a given st. line,* CE, *drawn from the same* • C, *to the st. line, which shall be on the opp. side of the perp.*



I. *The perp.* CD *is the least line from* C *to* AB ; CE > CD, CF > CE *and* CG > CF.

D.1	H	∵ ∠ CDE is a rt. ∠, ∴ ∠ CED < a rt. ∠ ;
2	H. & Sup.	If not, in △ CDE, ∠s CDE + CED ≮ two rt. ∠s ;
3	17, I.	which is impossible.
4	D. & 19, I.	∴ ∠ CDE > ∠ CED ∴ CE > CD.
5	D. 4.	Also ∵ ∠ CED < a rt. ∠ ∴ ∠ CEF > a rt. ∠.
6	*Sim.*	and in like manner ∠ CFD < a rt. ∠.
7	19, I.	∴ ∠ CEF > ∠ CFE, and CF > CE.
8	*Sim.*	So also CG > CF ;
9	Conc.	Hence the perp. CD is the least ; CE > CD, CF > CE, & CG > CF.

II. *From* C *only one st. line,* CK = CE *a given st. line from* C *to* AB *on the opp. side of the perp.* CD.

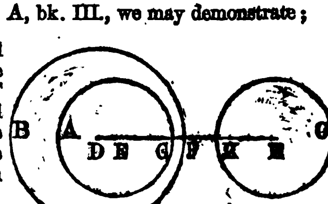
C.1	23, I.	At C in DC make ∠ DCH = ∠ DCE ;
2	Pst. 2.	and let CH cut AD in K ;
3	Conc.	Then CK = CE.
D.1	Ax.11, C.1	∵ rt. ∠ CDK = rt. ∠ CDE, ∠ DCK = ∠ DCE, and CD com.
2	26, I.	∴ CK = CE.
3	Remk.	Also no line from C to AB, but CK = CE ;
4	Sup.	If there can be, let it be CL ;
5	H.D.2 Ax.1	Then ∵ CL = CE & CK = CE ∴ CL = CK ; *i. e.* CL, more remote from CD than CK, equals CK ;
6	Case I.	which is impossible.
7	Conc.	Hence from C only one st. line to AB, namely CK = CE on the opp. side of perp. CD.

SCH. 3.—"Thys Proposition," says the Translator of 1570, "is called commonly in old bookes amongest the barbarous *Cauda Pauonis,* that is, the Peacockes taile."—Fol. 88.

USE AND APP.—I. The Proposition 8 Bk. III. is employed to show, that if a tangent and secant be drawn to the same point, the tang. < sec., but > external part of the secant : Thus in fig. to Prop. 8, let DH represent a tang. and MD a sec. to arc HG, or ∠ HMG, then DH < DM but > DG.

II. By aid of Props. 7 & 8, and Ax. A, bk. III., we may demonstrate ;

1st. When one circle A is contained within another, B, without touching it, the distance between the centres, DE < (DF ∾ EG), the difference of the radii : and *conversely*, when the distance between the centres, DE < (DF ∾ EG), the difference of the radii, the lesser circle will be within the greater without meeting it.



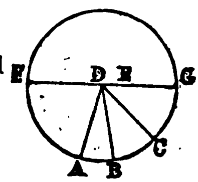
2nd. When two circles, B, C, lie each without the other and do not meet the distance between the centres, DH > (DF + HK,) the sum of the radii, and *conversely*, when the distance between the centres, DH > (DF + HK,) the sum of the radii, the circles lie each without the other and do not meet.

Prop. 9.—Theor.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Con. Pst. 1 & 2.

Dem. Def. 17, I. A diam. of a $\odot$ is any st. line through the cen., and terminated both ways by the $\odot$ce.—7, III.



E.1	Hyp.1	In $\odot$ ABC let • D be taken,
2	„ 2	& from D to $\odot$ce more than two equal lines, as DA = DB = DC;
3	Conc.	then • D shall be the cen. of the $\odot$.

Sup. *For if not, let* • E *be the centre.*

C.	Pst. 1 & 2.	Join DE, & prod. it to the $\odot$ce in F & G.
D.1	Def. 17, I.	∵ FG is the diam., & ∵ in FG is taken • D not the cen.
2	7, III.	∴ DG greatest line from D to $\odot$ce, DC > DB, and DB > DA;
3	H. *ad imp.*	but they are also equal;—an impossibility;
4	Conc.	∴ • E not the cen. of $\odot$ ABC.
5	*Sim.*	So no point but D the cen.
6	Conc.	∴ • D is the cen.
7	Rec.	∵ *If a point be taken, &c.* Q. E. D.

Cor.—1. *From any other point than the centre only two equal st. lines can be drawn to the* $\odot$*ce, whether the point* (7, III.) *be within, or* (8, III.) *without the circle.*

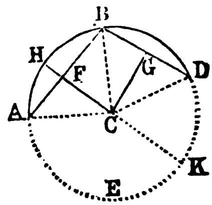
Cor.—2. *From three points given not in the same st. line the circumference of the circle may be found.*

SCH.—This Prop. gives the *criterion* for determining the centre of a circle; it is, that from a point, supposed to be the centre, to the circumference more than two points in the circumference shall be equally distant from the centre.

USE and APP.—By this Proposition the Problems may be demonstrated; 1st, *To draw a circle through three given points,* A, B, D; 2nd, *To find the centre of a given circle* ABDE; and 3rd, *To determine the centre of* ABD, *an arc of a circle.* The demonstration of the first, is equivalent to the demonstration of the other two.

C. 1	Pst. 1. 10, I.	Join ·s A, B, D and bis. AB and BD;
2	11, I.	at F and G raise perps. FC, GC meeting in C;
3	Psts. 1. & 3.	join CA, CB, CD and with either as rad. from C desc. a ⊙
4	Sol.	the ⊙ passes through A, B and D.
D1.	C. 1 & 2	∵ AF = FB, FC com. and ∠ AFC = ∠ BFC,
2	4, I.	∴ AC = CB
3	*Sim*	In like manner CD = CB
4	Ax. 1. & 9. III.	∴ CA = CB = CD & ∴ C cen. of ⊙ through ·s A, BD,

Q. E. F.



PROP. 10.—THEOR.

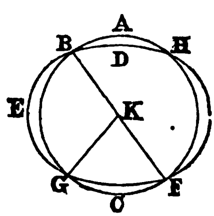
One circumference of a circle cannot cut another in more than two points.

CON. 1, III. Pst. 1.

DEM. Def. 15, I. 9, III.
5. III. If two circles cut one another they shall not have the same centre.

SUP.—*If possible let* ⊙ce FAB *cut* ⊙ce DEF *in more than two* ·s, *as in* B, G, F.

C.	1, III. Pst. 1	Take K cen. of ABC, and join KB, KG, KF
D.1	C. Def. 15, I.	∵ K cen. of ⊙ ABC ∴ KB = KG = KF:
2	C.	and ∵ within ⊙ DEF there fall from K to ⊙ more than two equal st. lines.



D.3	9, III.	∴ · K is cen. of $\odot$ DEF;
4	C.	but · K also cen. of $\odot$ ABC;
5	Conc.	∴ the same · is cen. of two $\odot$s cutting each other;
6	5, III.	which is impossible;
7	Rec.	therefore, *One circumference of a circle, &c.*

Q. E. D.

Sch.—"Two circles cannot have more than two points in common;" if they coincide in three points they will coincide in every point; or, "only one circle can be drawn through three given points."

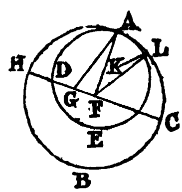
Prop. 11.—Theor.

If one circle touch another internally in any point, the straight line which joins their centres being produced shall pass through the point of contact.

Con. Psts. 1, 2.

Dem. 20, I. Def. 15, I. Ax. 5.

E.1	Hyp. 1.	Let $\odot$ ADE touch $\odot$ ABC internally in A,
2	„ 2.	and let F be cen. of $\odot$ ABC, G cen. of $\odot$ ADE;
3	Conc.	then the st. line joining F, G, being produced passes through A, the · of contact.



Sup.—*If* FG *produced do not pass through* A, *let it, if possible, fall as* FGDH.

C.	Pst. 1.	Join AF and AG.
D.1	20, I. Def. 15, I.	∵ FG + GA > FA, but FA = FH,
2		∴ FG + GA > FH,
3	*Sub.* Ax. 5.	take away com. pt. FG ∴ rem. AG > rem. GH;

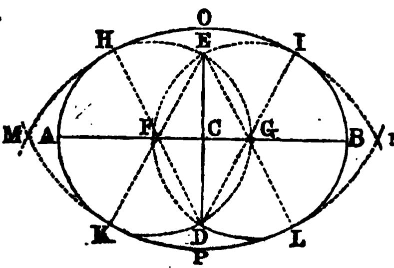
D.4	Def. 15, I.	but $AG = GD \therefore GD > GH$;
5	Remk. *ad imp.*	*i. e.* the less > the greater;—an impossibility.
6	Conc.	$\therefore$ FG joining F and G, being produced cannot fall except upon • A;
7	Remk.	*i.e.* FG prod. must pass through A, the • of contact.
8	Rec.	Therefore, *If one circle touch, &c.*

Q. E. D.

SCH.—When the distance, FK, between F and K, the centres of the two circles ADE and ABC is equal to the difference of the radii, AF and AK, the circles touch *internally*. For

C.	Pst. 1.	Take L, a • in ⊙ ADE, and join KL, FL;
D.1	20, I. Def. 15, I.	then $\because$ in △ FKL, $FL < FK + KL$, but $KA = KL$;
2		$\therefore FL < FK + KA$ *i. e.* $< FA$ $\therefore$ L within ⊙ ABC.
3	*Sim.*	So all other • s in ⊙ ADE except • A lie within ⊙ ABC.
4	Conc.	*When the distance between the centres, &c.*

USE AND APP.—I. By aid of this proposition *an oval may be described on any given major axis, as* AB.

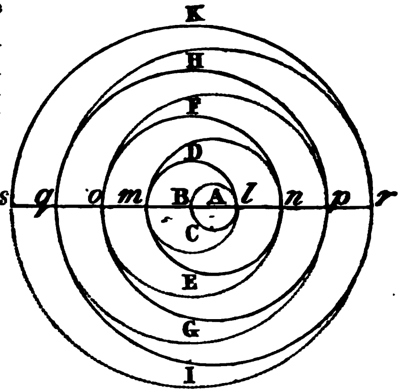


C.1	10, I. 11, 1.	Bis. AB in C by a perp. CD, CE
2	Remk.	the minor axis is OP, *i. e.* DE produced.
3	Pst. 3	If on C with rad. CA or CB a line revolve a circle will be traced;
4	Use 2, 34, I.	but div. AB into three eq. pts. $AF = FG = GB$;
5	Pst. 3.	from F and G with FA and GB desc. ⊙s AEGD and BEFD;
6	Pst. 1, 2.	from • s of inters. D, E, draw DE, DG, EF, EG, prod. to H, I, K, L;
7	Pst. 3.	next from D with rad. DH or DI desc. arc MHON,
8	Pst. 3.	and from E with rad. EK or EL desc. arc MKPN;
9	11, III.	the arcs AH, HI, IB, BL, LK, and KA touch in • s H, I, K, L,
10	Sol.	and the curve AHCIBLPKA will form an oval.

If the major axis be divided into four or more equal parts, by a similar method ovals more elongated, or with the minor axis in less proportion to the major, may be described.

N.B.—The oval thus described is only an approach to the true ellipse, the method being *practically* useful, not theoretically correct.

II. It is on the same principle that *a Spiral is described*, by successive semicircles taken alternately from two common centres A and B; for the line AB which joins them being produced, passes through the points of contact of the successive semicircles, *l, m, n, o, p, q, r, s*.



From A, the eye of the spiral, with rad. AB describe the semicircle C; from B, with B*l*, the semic. D; from A, with A*m*, semic. E; from B, with B*n*, semic. F; from A, with A*o*, G; from B, with B*p*, H; from A, with A*q*, I; and from B, with B*r*, K. The spiral may be continued to any extent in the same way.

A *Spiral* is a curve line making revolutions round the centre or eye of the curve, which do not return into themselves as the revolutions of a circle do. A *Plane Spiral* is generated by a continually increasing radius, and according to the law or rate of increase spirals differ in their curves. Our older English writers considered the circle and the ellipse to be spirals; but they are excluded by the above definition, inasmuch as their revolutions return into themselves, or to the very point from which they started.

Besides the above, the principal plane Spirals are, the Spiral of Archimedes or Conon,—the Hyperbolic or Reciprocal Spiral,—the Lituus, and the Logarithmic Spiral; but they can only be noticed here by way of definition. 1st, When from a given point any number of lines are drawn forming equal angles at that point, and the length of each line increases in succession by an equal quantity, the curve which passes through these points is named *the Spiral of Archimedes*. 2nd, The *Hyperbolic* or *reciprocal Spiral* is a curve passing through the extremities of any number of arcs of circles of equal length measured from a given st. line. 3rd, The *Lituus*, so named from the crooked staff of the Roman augurs, is a Spiral to be thus described;—"Let a variable circular sector always have its centre at one fixed point, and one of its terminal radii in a given direction. Let the area of the sector always remain the same, then the extremity of the other terminal radius as it revolves describes the Lituus." 4th, The *Logarithmic Spiral*, in which the radii make equal angles, and the spiral cuts them all at an equal angle, the length of the successive radii increasing in geometrical progression.

It may be observed that curves are infinite in variety, though only about thirty have received specific names. The Parabola, Hyperbola, Cycloid, Watt's Parallel motion curve, &c., are among them; but it would be out of place to explain them here.

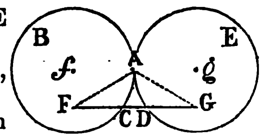
Prop. 12.—Theor.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point of contact.

Con. Pst. 1.

Dem. Ax. 2. 9. Def. 15, I. 20, I.

E.1	Hyp. 1.	Let the two ⊙s ABC, ADE touch extern. in A;
2	„ 2.	and let F be the cen. of ABC, G of ADE;
3	Conc.	then FG shall pass through A, the • of contact.

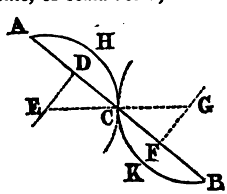


Sup. If not, let it pass through C and D.

C.	Pst. 1.	Join FA and GA.
D.1	H.2. Def.15,I.	∵ F is cen. of ⊙ ABC ∴ $FA = FC$;
2	H.2. Def.15,I.	and ∵ G is cen. of ⊙ ADE ∴ $GA = GD$;
3	Ax. 2. 9.	∴ $FA + GA = FC + GD$, and ∴ $FG > FC + DG$.
4	20, I. *ad imp.*	But $FG < FA + AG$; which is impossible.
5	Conc.	∴ the st. line FG cannot pass except through A the • of contact; *i. e.* FG must pass through A.
6	Rec.	Therefore, *If two circles touch, &c.* Q. E. D.

Sch.—Should fg, the distance between the centres of two circles, be equal to $fA + Ag$, the sum of the radii, the circles touch each other externally.

Use and App.—The drawing of a *Serpentine Line*, or *coma recta*, between two given points, as A, B, depends on this 12th Prop. For join AB, and bisect it in C; again bis. AC by perp. DE and BC by perp. FG; make $DE = FG$; and from E with rad. EC desc. arc AHC, and from G with rad. GC, the arc CKB; the arcs touch in the • C, and the two arcs form the serpentine AHCKB; which might be continued to any extent by following up the same process.



Prop. 13.—Theor.

One circle cannot touch another in more points than one, whether it touches it on the inside or the outside.

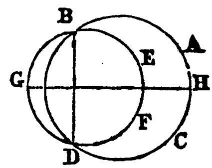
Con. Pst. 1, 11, I.

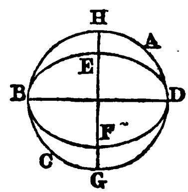
Dem. 2, III. 11, III. Cor. 1, III.

E.1	Hyp.	If possible let ⊙ EBF touch ⊙ ABC in more • s than one.

I.—*On the inside in the* • s B, D.

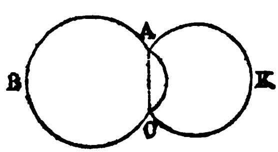
C.	Pst.1.11,I.	Join B D, and draw G H bisecting B D ⊥.
D.1	H. 2, III.	∵ • s B, D are in the ⊙ce of each ⊙ ∴ B D falls within them;
2	Cor.1,III.	∴ their centres are in G H which bis. B D ⊥;
3	11, III.	∴ G H passes through the • of contact;
4	H.	but • s B, D, are without the st. line G H,
5	Conc.	∴ G H does not pass though the • of contact, which is absurd.
6	Rec.	∴ one ⊙ cannot touch another internally in more • s than one.





II.—*If possible, let* ⊙ ACK *touch* ⊙ ABC *externally in* A *and* C.

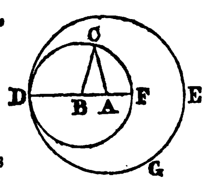
C.	Pst. 1.	Join A C.
D.1	Hyp.	∵ A & C are in the ⊙ce of the ⊙ A C K,
2	2, III.	∴ A C which joins them falls within ⊙ A C K;
3	H.	But ⊙ A C K is without ⊙ A B C, ∴ A C is without ⊙ A B C;



D.4	H.	but ∵ A & C are • s in the ⊙ce of the ⊙ ABC,
5	2, III.	∴ AC must also be within the same ⊙; an absurdity;
6	Conc.e	∴ one ⊙ cannot touch another on the *outside* in more • s than one.
7	Case I.	And ⊙s cannot touch on the *inside* in more • s than one;
8	Rec.	Therefore *one circle cannot touch, &c.* Q.E.D.

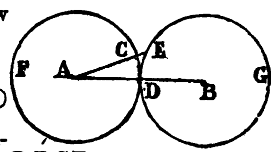
SCH.—It has been by assuming two points of contact between circles, and by showing the assumption to be impossible, that the proposition has been proved, that two circles cannot touch in more parts than one. But for this *indirect* method of demonstration, it is better to substitute the *direct* method; thus

1st. If the circles touch *internally*, as at D, each point in the ⊙ce of the less ⊙, except the common point of contact, D, through which the line, AB, joining their centres, A, B, passes, must be within the ⊙ce of the other ⊙.



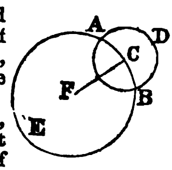
C.	Pst. 1.	For take C, a • on the ⊙ce of DCF, & join centres & C.
D.1	7, III.	Here $AD > AC$.
2	D. 1.	And ∵ $AC < AD$, the rad. of ⊙ DEG;
3	Ax. A. III.	∴ the • C is within ⊙ DEG;
4	*Sim.*	So every •, except D, of ⊙ DCF is within the ⊙ DEG.

2nd. If the ⊙s DCF and DEG touch *externally*, every point of the one, except the common point of contact is without the other.



C.	Pst. 1.	Join AB, the centres, and draw ACE,
D.1	H. & 8, III.	∵ the ⊙s touch externally, ∴ $AE > AC$ or AD;
2	Ax. A, III.	∴ the • E lies out of the ⊙ DCF;
3	*Sim.*	thus every • of ⊙ DEG, except the • D, lies out of the ⊙ DCF.

USE and APP.—The four Propositions, 10, 11, 12 and 13, are employed by astronomers to explain the motion of the Planets in Epicycles. An *Epicycle* is a circle, ABD, the centre of which, C, is carried round upon another circle ABE.



PTOLEMY of Alexandria, the celebrated astronomer, A.D. 139, in explaining the motions of the planets on what is called the Ptolemaic System, employs the theory of Epicycles: but the common notion is erroneous that this use of the Epicycle is peculiar to the Ptolemaic Astronomy: "the modern astronomer to this day resolves the same motions into epicyclic ones. When the latter expresses a result by series of sines and cosines (especially when the angle is a mean motion or a multiple of it) he uses epicycles; and for one which Ptolemy scribbled on the heavens, to use Milton's phrase, he scribbles twenty."—A. De M. *Gk. and Rom. Biography*, Vol. III., p. 576.

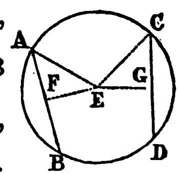
PROP. 14.—THEOR.

Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another.

CON. 1, III.
12, I. To draw a perp. to a st. line from a point without it. Pst. 1.

DEM. 3, III.
Ax. 7. Things that are halves of the same thing, or of equals, are equal.
Def. 15, I.
47, I. In any rt. ∠d △ the square on the side opp. to the rt. ∠ shall be equal to the squares on the sides including the rt. ∠.
Axs. 1, 3. Def. 4, III.
Ax. 6. The doubles of the same thing, or of equal things, are equal.

E.1	Hyp.	I. Let AB = CD in ⊙ ABCD;
2	Conc.	then AB & CD shall be equally distant from cen. E.
C.1	1, III.	Find E the cen. of ⊙ ABCD.
2	12,I.Pst.1	from E draw EF, EG ⊥ AB, CD & join EA, EC.
D.1	C. 2	∵ EF through cen. E cuts AB not through cen. at rt. ∠s;
2	3, III.	∴ EF bis. AB, & AF = FB, and AB = 2AF;
3	*Sim.*	For the same reason CD = 2CG;
4	H. & Ax.7	but AB = CD ∴ AF = CG.
5	Def. 15, I.	And ∵ AE = EC ∴ $AE^2 = EC^2$;
6	C.2 & 47, I.	but ∠ AFE being a rt. ∠ ∴ $AF^2 + FE^2 = AE^2$;
7	*Sim.*	So $EG^2 + GC^2 = EC^2$;
8	Ax. 1.	∴ $AF^2 + FE^2 = EG^2 + GC^2$;
9	D. 4.	but AF = CG ∴ $AF^2 = CG^2$;
10	Ax. 3.	∴ rem. FE^2 = rem. EG^2, & ∴ FE = EG;
11	Def.4, III.	but lines are equally dist. from cen. when the perps. to them from the cen. are equal,
12	Conc.	∴ AB and CD are equally dist. from E.



E.1	H.Df.4,III	II. Let AB & CD be equally dist. from E; *i. e.* FE = EG;
2	Conc.	then AB = CD.
C.	*Sim.*	Let the same construction be made.
D.1	C.&3,III.	∵ EF bis. AB and EG bis. CD; ∴ AB = 2AF and CD = 2CG;
2	47, I.	and $EF^2 + FA^2 = EG^2 + GC^2$;
3	H.	and ∵ FE = EG, ∴ $FE^2 = EG^2$;
4	Ax. 3.	∴ rem. AF^2 = rem. CG^2, and ∴ AF = CG.
5	D.1. Ax.6.	But AB = 2AF, and CD = 2CG; ∴ AB = CD.
6	Rec.	Therefore, *Equal st. lines are equally, &c.* Q. E. D.

SCH.—A principle employed in this and the next proposition is—if two quantities $A + B = C + D$, then if $A = C$, $B = D$; if $A > C$, $B < D$; if $A < C$, $B > D$.

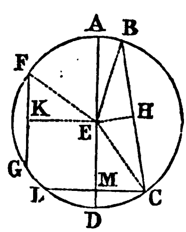
PROP. 15.—THEOR.

The diameter is the greatest straight line in a circle; and of all others that which is nearer to the centre is always greater than one more remote: and the greater is nearer to the centre than the less.

CON. 12, I. Pst. 1.

DEM. Def. 15, I. Ax. 1. 20, I. 14, III. 47, I. Ax. 5.
Def. 5, III. The st. line on which the gr. perp. from the cen. falls is farther from the cen.

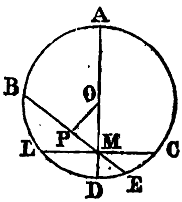
E.1	Hyp. 1.	I. Let ABCD be a ⊙, AD its diam., and E its cen.
2	„ 2	and let BC be nearer to E than FG is;
3	Conc.	then AD > any line BC not a diam., and BC > FG.
C.1	12, I. Pst. 1.	From E draw EH, EK perp. to BC, FG; and join EB, EC, EF.



D.1	Def.15,I.Ax.2.	$\because$ AE = EB, and ED = EC; $\therefore$ AD = EB + EC;
2	20, I.	but $\because$ EB + EC > BC, $\therefore$ AD > BC.
3	H. 2.	And $\because$ BC is nearer cen. E than FG;
4	Def. 5, III.	$\therefore$ EH < EK;
5	14, III. D.	but BC = 2BH, FG = 2FK,
6	47, I.	and $EH^2 + HB^2 = EK^2 + KF^2$;
7	D. 3.	and $\because$ EH < EK $\therefore$ $EH^2 < EK^2$;
8	Conc.	$\therefore$ $BH^2 > FK^2$ and BH > FK; and $\therefore$ BC > FG.
E.1	H.	II. Next, let BC > FG;
2	Conc.	then BC is nearer cen. E than FG is; *i. e.* EH < FK.
D.1	H.&D.7. Case I.	$\because$ BC > FG, BH > KF, and $BH^2 > FK^2$;
2	D. 5. Case I.	and $\because$ $BH^2 + HE^2 = EK^2 + KF^2$;
3	Ax. 5.	$\therefore$ rem. $HE^2 < EK^2$, and HE < EK;
4	Def. 5, III.	and $\therefore$ BC is nearer the cen. E than FG is.
5	Rec.	$\therefore$ *The diam. is the greatest st. line, &c.*

Q. E. D.

Sch. 1.—In a circle the *longest* chord is the diam., as AD; the *shortest*, through a given point, as M, is that, LC, which is perp. to the longest.



Through M, a given · draw a diam. AD, and any other chord, as BE; and to AD, through M, a perp. LC also a chord; and from O the cen. of the ⨀ draw OP perp. to BE.

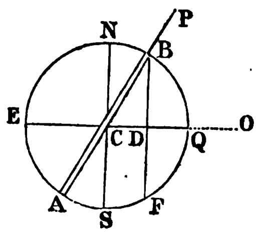
By 47, I. OM > OP; and, 15, III., BE > LC; and the same being true of any other chord through M, $\therefore$ LC at rt. ∠s to AD is the *least* chord, and, 15, III., AD the *longest*, through · M.

2. The *less* the ∠ which a given chord through a given ·, as M, makes with the diam. the *greater* will the chord be;

For, it is evident, that as the ∠AMB diminishes, the perp. OP, marking the distance of the chord from the cen., will also diminish, and the chord increase. Lardner's Euclid, p. 104.

Use and App.—I. The 14th and 15th Props. were employed by Theodosius to demonstrate that in a Sphere the centres of the least circles on the sphere are the most distant from the centre of the sphere itself; or, in other words, that the circles of latitude diminish as the poles of the earth are approached.

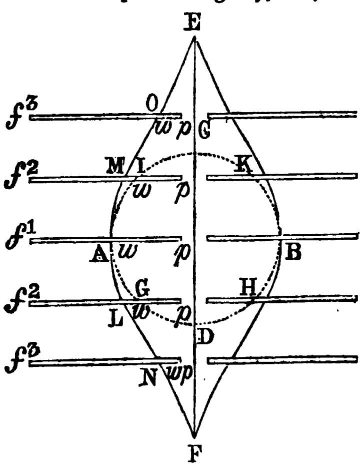
II. In the *Astrolabe*, a circular instrument for *taking the stars*, *i. e.* for observations on the heavenly bodies, the same propositions are serviceable. HIPPARCHUS, the greatest of Greek astronomers, B.C. 160, was probably the first who constructed an instrument of this kind. Its general nature may be seen from the following representation, in which the approach of the chord BF to the diam. NS may be measured, either by the perp. CD, or the arc BQ.



Let NESQ be a circle fixed to one position, vertical or horizontal; and let AB be a line or tube moveable round the cen. C; through the tube, along the line AB, any object, as P, may be seen; and, by turning the tube round its centre C, another object, as O, may also be seen; the ∠ QCB, subtended by the two objects is their *angular distance*, which may be measured in degrees when the circle NESQ is graduated.

If EQ represent the equator, and if the plane of the circle ESQN pass through the poles N and S; then the ∠ BCN is the north *polar distance*, and the ∠ BCQ, the *declination* of the star or planet P. Again, if the circle ESQN be fixed in the plane of the equator, and EQ point to the vernal equinox at the instant that the tube, AB, points to the star P, then the ∠ BCQ will be the right ascension of the star.

III. It has been said that ARISTOTLE, B.C. 340, propounded a question in mechanics similar to the following:—"At what part of a galley, EF, with rounded sides, EAF, EBF, does an oar, handled from a seat, or station, just above the line of the keel, EF, produce the greatest effect in moving the galley?"



If we examine the conditions of the rounded sides, we shall find that AB, the diam. of the ⊙ ACBD, is the greatest width; then, by 15, III., AB being the greatest line in the circle, all other chords, GH, IK, are less than AB. When the oar is applied at A, the leverage pf^1 of the rower, in proportion to wf^1, the leverage of the weight or resistance, is greater than when the oar is applied at L or at M; where again it is greater than when applied at N or O. The best position therefore for the oars is in the line of the greatest width, AB.

PROP. 16.—THEOR.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle, and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

CON. Pst. 1. 12, I.

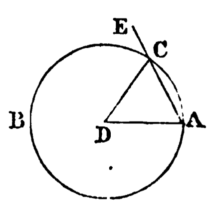
DEM. Def. 15, I.
5, I. The $\angle$s at the base of an isosc. $\triangle$ are equal.
17, I. Any two $\angle$s of a $\triangle$ are together less than two rt. $\angle$s.
Ax. 9. 19, I. The gr. $\angle$ of any $\triangle$ shall be subtended by the gr. side.
Def. 2, III. A st. line is said to touch a $\odot$, when it meets the $\odot$, and being produced, does not cut it.
2, III. If any two · s be taken in the $\odot$ce of a $\odot$, the st. line which joins them shall fall within the $\odot$.

E.1	Hyp. 1	Let ABC be a $\odot$, fig. 3, D the cen., and AB the diam.;
2	„ 2	and the st. line AE be $\perp$ to AB, at its extremity A;
3	Conc. 1	then AE shall fall without the $\odot$ ABC;
4	„ 2	& no st. line can be drawn from A, between AE and the $\odot$ce, unless it cuts the $\odot$.

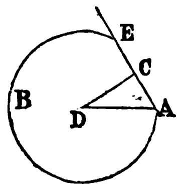
I. *The st. line* AE, *fig.* 3, *shall fall without the* $\odot$ ABC.

SUP. 1 & 2.—If not, it must fall either *within* or *on* the $\odot$ce.

	Sup. 1.	1st, Let AE fall *within* the $\odot$.
C.	Pst. 1.	Join D and C, the · where AE cuts the $\odot$ce.
D.1	Def. 15, I. 5, I.	$\because$ DA = DC $\therefore$ $\angle$ DAC = $\angle$ DCA;
2	H. 2.	but $\angle$ DAC is a rt. $\angle$, $\therefore$ $\angle$ DCA is a rt. $\angle$.
3	D.1 & 2, 17, I.	$\therefore$ in $\triangle$ ADC, the $\angle$s DAC & DCA = 2 rt. $\angle$s; which is impossible
4	Conc.	$\therefore$ AE if $\perp$ to DA does not fall *within* the $\odot$.



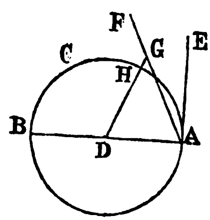
	Sup. 2.	2nd, If possible let AE fall *on* the $\odot$ce.
C.	Pst. 1.	In $\odot$ce take a $\cdot$ C and join DC.
D.1	D.1,2&3, 1st.	As before in $\triangle$ ADC, $\angle$s DCA & DAC = 2 rt. $\angle$s.
2	17, I.	which is impossible;
3	Conc.	$\therefore$ AE cannot fall *on* the $\odot$ce.



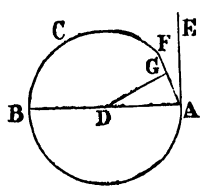
II. *No st. line can be drawn from* A, *between* AE *and the* $\odot$*ce, unless it cuts the circle.*

Sup. 1 and 2.—If such a line can be drawn, as AF, it falls either *without* or *on* the $\odot$.

	Sup. 1.	1st. Let AE fall *without* the $\odot$ ABC.
C.	12, I.	From D sup. DG $\perp$ AF, and meeting $\odot$ce in H.
D.1	Ax. 9. H. 2.	$\because$ $\angle$ DAG < $\angle$ DAE, but $\angle$ DAE a rt. $\angle$;
2		$\therefore$ $\angle$ DAG < a rt. $\angle$.
3	C.	But $\angle$ DGA is a rt. $\angle$, $\therefore$ $\angle$ DGA > $\angle$ DAG;
4	19, I.	and $\therefore$ DA > DG;
5	Def.15,I.Ax.9.	now DA = DH, $\therefore$ DH > DG, a part > the whole; an impossibility;
6	Conc.	$\therefore$ AF cannot fall *without* the circle.



	Sup. 2.	2nd. If possible then, let AF fall *on* the $\odot$ce.
C.	12, I.	Draw DG $\perp$ AF, G being in the supposed $\odot$ce.
D.1	D.1,2,3CaseII.	Then as before DA > DG;
2	Def. 15,I.	But DA = DG; which is impossible;
3	Conc.	$\therefore$ AF cannot fall *on* the $\odot$ce.
4	Rec.	Hence, *The st. line drawn at rt.* $\angle$*s, &c.*



Q. E. D.

Cor. I.—*If a st. line,* AE, *be drawn at rt.* $\angle$*s to any diam. of a* $\odot$, AB, *from its extremity,* A, *it shall touch the* $\odot$ *at the*

extremity: and a st. line touching the $\odot$ at one • shall touch it at no other point.

D.1	Sup.&2,III.	For, if AE cut the $\odot$, part would fall within the $\odot$;
2	16, III. pt. I.	which has been proved to be impossible:
3	Def. 2, III.	$\therefore$ AE touches the $\odot$ at A, the extremity of BA.
4	Remk.	And AE touches the $\odot$ at no other point except A;
5	Sup.&2,III.	for if it could, as before, part would fall within the $\odot$.
6	16, III. pt. I.	which has been proved to be impossible.

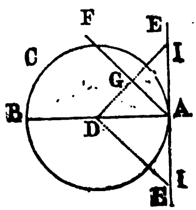
Cor. II.—*By* 28, I. *st. lines at rt.* $\angle$*s to the extremities of the same diam. are parallel.*

Cor. III.—*Tangents to a circle from the same point are equal.*

Sch.—1. Thus, a Tangent to a circle is a rt. line perpendicularly raised on the extremity of a diam., or of a radius.

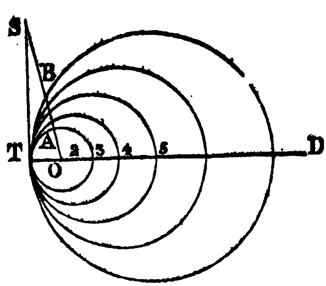
2. The 16th Prop. might have been proved *directly*; thus

C.	Pst. 1.	I. From D draw any line DI to meet AE;
D.1	H. 32, I.	$\because$ DAI is a rt. $\angle$. $\therefore$ $\angle$DIA $<$ $\angle$DAI;
2	19, I. Ax. A.	& $\because$ DA $<$ DI; $\therefore$ the • I is out of the $\odot$.
3	*Sim.*	So, we may prove the same of every • in AE except • A.
C.	Psts. 1,2,I.	II. Let AF make $\angle$ FAD acute; and DG $\perp$ AF.
D.1	C. 32, I.	$\because$ DGA is a rt. $\angle$, $\therefore$ $\angle$GAD $<$ $\angle$DGA;
2	19, I. Ax. A.	and $\because$ DG $<$ DA, $\therefore$ G is within the $\odot$, and AF cuts it.
3	Conc.	$\therefore$ the line EAE is a tang. to the $\odot$, and meets it only in the • A.



Use. and App.—I. The *infinite divisibility* of linear magnitude may be proved by this proposition.

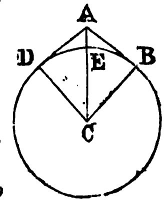
If in TD, or TD produced, points, O, 2, 3, 4, 5, be taken as centres of circles, to all of which TS shall be a com. tang.; then each circle as it cuts AS shall approach nearer to S, as B; but no circle shall ever pass through the point S, inasmuch as TS being a com. tang., the circles cannot touch TS except in the point T; for if they did, a curved line and a st. line would coincide.



II. Since by Prop. 16, III., the tang. A B is perp. to CB, the rad. of the circle, we can, by 47, I. compute the *distances and heights* of objects, on the earth's surface, when they are situated on the verge of the natural horizon; thus, 1st, if A B the distance of the verge of the horizon from the elevation A E be required,—$AB = \sqrt{CA^2 \text{ dist. of A from earth's centre}; - CB^2 \text{ earth's rad.}}$; where $CA^2 = (CE + EA)^2$.

2nd, If A E be the height of an object just visible from the verge of the horizon B or D; then $AE = CA - CE$ or CB; where $CA = \sqrt{AB^2 + BC^2}$;

Ex. 1. *Given* $AE = 2$ *miles, the height of the peak of Teneriffe, at what distance will its summit be seen?*



Here CB or CE = 4000 miles;
& CA = CE + AE = 4000 + 2 = 4002 miles.
Then $4002^2 - 4000^2 = AB^2$.
$\therefore 16016004 - 16000000 = 16004$;
& $\sqrt{16004} = 126$ miles = A B distance at which A is visible.

Ex. 2. *A meteor* A *was seen over a distance from* B *to* D *of* 500 *miles; required* E A, *its height above the earth.*

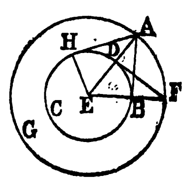
Here $AE = AC - EC$ or $= CB$; and $\sqrt{CB^2 + AB^2} = AC$.
Or, $\sqrt{4000^2 + 250^2} = \sqrt{16062500} = 4007 \cdot 84$ miles;
and $4007 \cdot 84 - 4000 = 7 \cdot 84$ miles, height of the meteor.

Prop. 17.—Prob.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

Sol. 1, III. Psts. 1, 3. 11, I.

Dem. Def. 15, I. 4, I. Ax. 1. Cor. 16, III.

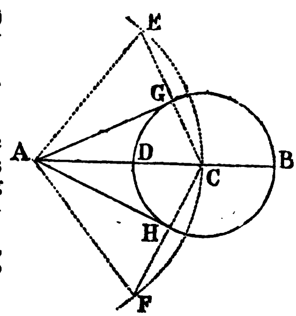


E.1	Dat.	I. Let A be a given • without the given ⊙ BCD;
2	Quæs.	from A to draw a tang. to the ⊙ BCD.
C.1	1, III. Pst. 1.	Find E cen. of ⊙ BCD, and join A E;
2	Pst. 3.	from E with E A desc., the ⊙ AFG;
3	11, I. Pst. 1.	at D draw the perp. D F; and join E B F and A B;
4	Sol.	then A B shall touch the ⊙ BCD in B.

D.1	C.1,2. Def.15,I.	∵ E is the com. cen. of ⊙s BDC, AFG; ∴ EA = EF, and ED = EB;
2	D. 1 and C.	and ∵ AE, EB = EF, ED, and ∠ E com. to △s AEB, FED;
3	4, I.	∴ DF = AB, △ FED = △ AEB, & ∠ EBA = ∠ EDF.
4	C. 3. Ax. 1.	but ∠ EDF is a rt. ∠, ∴ ∠ EBA is a rt. ∠;
5	Remk.	now EB is a line from the cen to the ⊙ce;
6	D.4. Cor.16, III.	∴ AB touches the ⊙ in B, and is drawn from A.
E.1	Dat.	II. Let D be the given · *in* the given ⊙ce BCD;
2	Quæs.	from D to draw a tang. to the ⊙ BCD.
C.1	1, III. Pst. 1.	Find the centre E, and join DE;
2	11, I.	from D draw DF perp. to DE;
3	Sol.	then DF is the tang. to the ⊙.
D.1	C. 2.	∵ DF is perp. at D to the rad. ED;
2	Cor. 16, III.	∴ DF touches the ⊙ BCD at the given · D. Q. E. F.

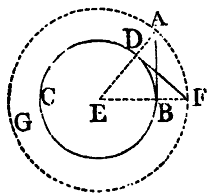
SCH.—It is evident that from · A, out of the ⊙ BDG, two equal tangents, AG and AH, may be drawn.

C.1	1, III. Pst. 1, 2.	Find C the cen. of ⊙ BDG, and draw ACB;
2	Pst. 3.	from A with AC desc. arc ECF;
3	Pst. 3.	and with rad. DB, from C, cut arc ECF in E and F;
4	Pst. 1.	join CE, CF, cutting the ⊙ BDG in G and H;
5	Sol.	the lines AG, AH, will be the two equal tangents.



D.1	Def. 15, I. 3, III.	For, ∵ AC = AE and CG = GE, and AG com.
2	8, I.	∴ △ AGE = △ AGC, and ∠ AGE = ∠ AGC;
3	Def.10, I. Cor.16, III.	∴ AG ⊥ CE and tang. to CG a rad.
4	*Sim.*	So AH is the second tang. to the ⊙ BDG.
5	Def. 15, I. 5, I.	Also ∵ CG = CH, AC com. and ∠ GCA = ∠ HCA;
6	4, I.	∴ tang. AG = tang. AH.

USE AND APP.—I. *Practically* a tangent will be drawn, from a point A *out* of a circle DBC by aid of a ruler; and from a point D *on* the circumference, by joining the point and the centre, E, and at the point, making, DF, a perpendicular to the radius.



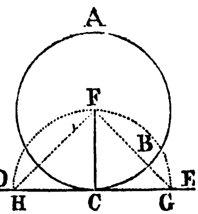
II. Tangent lines, as AB, FD, determined by secants EA, EF, are of very frequent use, especially in trigonometry; tables of the proportional or relative lengths of chords, natural lines, tangents, and secants are constructed by aid of Prop. 47, Book I.

PROP. 18.—THEOR.

If a st. line touches a circle, the st. line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

CON. 1, III. Pst. 1. 12, I.

DEM. 17, I. 19, I. Def. 15, I. Ax. 9.



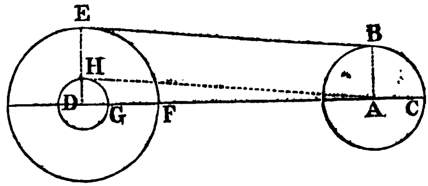
E.1	Hyp.	Let the st. line DE touch the ⊙ ABC in C;
2	1, III. Pst. 1.	take the cen. F, and draw the st. line FC;
3	Conc.	then FC shall be perp. to DE.

SUP.—*If not, from* F *draw* FBG *perp. to* DE.

D.1	C. 17, I.	∵ FGC is a rt ∠, ∴ GCF is an acute ∠;
2	19, I.	and ∵ to the gr. ∠ the gr. side is opp. ∴ FC > FG:
3	Def. 15, I. Ax. 9.	but FC = FB ∴ FB > FG, the less than the gr.; which is impossible;
4	Conc.	∴ FG is not perp. to DE.
5	*Sim.*	So, no line except FC is perp. to DE;
6	Conc.	∴ FC is perp. to DE.
7	Rec.	∴ *If a st. line touches, &c.* Q. E. D.

SCH.—Propositions 16 and 18 may be regarded as the *converse* of each other; in the one we prove, that a line perpendicular to the extremity of the radius is a tangent; in the other, that, if a line be a tangent to a circle, the radius from the point of contact will be perpendicular to it. Thus in Prop. 16, AE is proved to be perp. to AB, and in Prop. 18, FC perp. to CE: in the one case, the tang. is perp. to the rad.; in the other, the rad. is perp. to the tang.; two expressions for the same thing.

USE AND APP.—*To draw a tang. to each of two given* $\odot$*s* ABC, DEF.



C.1	Pst. 1. 3, I.	Join the centres A, D, and make FG = AC;
2	Pst. 3.	from D with rad. DG desc. $\odot$ DGH;
3	17, III.	and from A draw AH, tang. to $\odot$ DGH;
4	Pst. 1, 2.	join DH, and prod. it to E in $\odot$ DFE;
5	31, I. Pst. 1	from A draw AB ∥ DE, and join EB;
6	Sol.	then EB is tang. to each $\odot$, ABC and DEF.

D.1	C 3, 18, III	∵ AH is tang. in H to $\odot$ DGH, ∴ $\angle$ DHA is a rt. $\angle$;
2	13, I. C.	and $\angle$ EHA is a rt. $\angle$; and ABEH is a ▱;
3	C. & 46, I.	and ∵ $\angle$ EHA is a rt. $\angle$ ∴ all the $\angle$s in ABEH are rt. $\angle$s.
4	D. 3	then ∵ DE from cen. D, & AB from cen. A, meet BE at rt. $\angle$s.
5	Cor 16, III	∴ EB is tang. to both $\odot$s, ABC and DEF. Q. E. F.

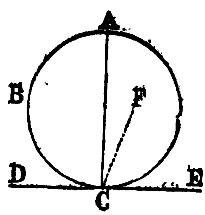
PROP. 19.—THEOR.

If a st. line touches a circle, and from the point of contact a st. line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

CON. Pst. 1.

DEM. 18, III. Ax. 11. All rt. $\angle$s. are equal to one another.—Ax. 9.

E.1	Hyp. 1	Let the st. line DE touch the $\odot$ ABC in C;
2	„ 2	& from C let CA be at rt. $\angle$s to DE;
3	Conc.	then the cen. of the $\odot$ shall be in CA.

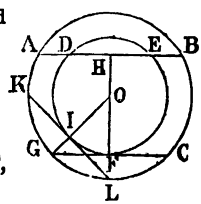


Sup.—*For if not, and it be possible, let* F *be the cen. and join* F C.

D.1	Sup.	then ∵ D E is a tang. and F C a rad. from the · of contact;
2	18, III.	∴ F C is perp. to D E, and ∠ F C E a rt. ∠;
3	H. 2, Ax.11	but A C E is a rt. ∠, ∴ ∠ F C E = ∠ A C E;
4	Remk. Ax.9.	*i. e.* the part = the whole, which is impossible.
5	Conc.	∴ the · F is not the cen. of the ⊙ A B C.
6	*Sim.*	And thus no other ·, except it be in C A, is the cen.;
7	Conc.	∴ the cen. of the ⊙ is in C A.
8	Rec.	Therefore, *If a st. line touches, &c.* Q. E. D.

Sch.—1. If in two concentric circles A B C, D E F, a chord AB of the greater meet the less, as in D and E, the parts A D and E B, intercepted between the two ⊙s are equal; and all chords of the greater ⊙, as G C, which touch the less, are bisected at the · s of contact, as F; and are equal.

C.	12, I.	From O the cen. draw O H ⊥ A B, and O F ⊥ G C.
D.1	3,III. Ax.3.	I. ∵ A H = B H and D H = E H ∴ A D = B E.
2	C. 3, III.	II. ∵ ∠ O F C = ∠ O F G; ∴ G F = F C.
3	*Sim.*	Thus all chords of ABC tangs. to DEF, are bisd. at the · of contact.
4	14, III.	and III. such chords are equal.



2. If any number of equal chords, G C, K L, be drawn in a circle, the *locus* of their points of bisection, as F, I, &c., is a circle of which the rad.^{2} = O G^2 *minus* G F^2.

Use and App.—I. In Optics the properties of tangent lines are employed, among other purposes, for determining the part of a globe which may be enlightened by a luminous body, as by a lighthouse, a volcano, a meteor, or the sun.

II. By a tangent line representing the limit or extent of vision, the diameter of the earth may be ascertained, as in Prop. 6, Book II.

III. Tangent lines, too, serve to explain the Theory of the Phases of the Moon, and were employed by Hipparchus, B.C. 160 to 145, to ascertain the distance of the sun.

IV. In Navigation the dip of the horizon corresponds with the tangent line from the point of observation; and were the tangent line to revolve round that point it would trace out the circle of the physical horizon. An approxi-

mate rule for ascertaining the distance of the verge of the natural horizon is to take the square root of the height of the spectator's eye in feet and multiply it by 1·3 ;—the product in feet will nearly give the distance, or rad. of the horizon, in miles ; thus from the mast of a ship 81 feet above the sea the horizon is 11·7 miles off; for $\sqrt{81} \times 1·3 = 9 \times 1·9 = 11·7$; or,

How distant is the tangent point from the highest peak of Teneriffe, which has an elevation of about 11,946 feet ?

$\sqrt{11,946} \times 1·3 = 109 \times 1·3 = 141·7$ miles.

V. In Dialling, also, or Gnomonics, tangents are employed for calculating the hour lines and ascertaining their exact position ; but Dialling is now comparatively of little importance.

Prop. 20.—Theor.

The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

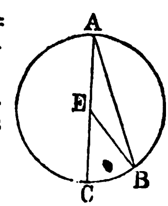
Con. Pst. 1 & 2.

Dem. Def. 15, I. 5, I.
32, I. In every △, if any side be produced, then the ext. ∠ = the two int. and opp. ∠s ; and the three int. ∠s = two rt. ∠s.

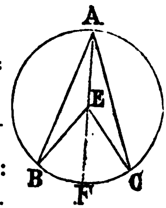
E.1	Hyp. 1	In a ⊙ ABC let E be the cen.; ∠ BEC the ∠ at the cen.
		and ∠ BAC the ∠ at the ⊙ce;
2	„ 2	and let ∠s BAC, BEC have for base the same arc BC;
3	Conc.	then the ∠ BEC = twice ∠ BAC.
C.	Pst. 1. 2.	Join AE and produce it to the ⊙ce F. (fig. 2.)

Case I. *Let* E *the cen. be* on AC, *one of the st. lines* AB, AC.

D.1	Def.15,I.5,I.	∵ EA = EB ∴ ∠ EAB = ∠ EBA; and ∠s EAB + EBA = 2 ∠ EAB;
2	C. 32. 1.	but ∵ in △ AEB, AE is prod. to C, ∴ ext. ∠ BEC = ∠s EAB + EBA
3	Conc.	∵ ∠ BEC = twice ∠ BAC.

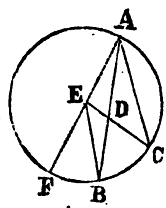


CASE II. *Let* E *the cen. be* within *the* ∠ BAC



D.1	Def.15,I.5,I.	∵ EA = EB, ∴ ∠ EAB = ∠ EBA,
2	Remk.	and ∠s EAB + EBA = twice ∠ EAB;
3	C. 32, I.	but ∠ BEF = ∠ EAB + ∠ EBA;
4	Conc.	∴ ∠ BEF = twice ∠ EAB:
5	*Sim.*	Thus also ∠ FEC = twice ∠ EAC;
6	Conc.	∴ the whole ∠ BEC = twice the whole ∠ BAC.

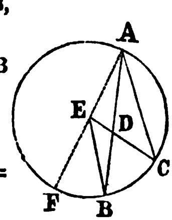
CASE III. *Lastly let* E *be* without *the* ∠ BAC.



D.1	32, I.	then ∵ ∠ FEC = 2 ∠ FAC, & ∠ FEB = 2 ∠ FAB;
2		∴ rem. ∠ BEC = twice rem. ∠ BAC.
3	Rec.	Therefore in every case, *the angle, &c.* Q. E. D

COR.—*Any angle, as* ∠ CAB, *at the circumference is measured by half the arc* CB, *on which it stands.*

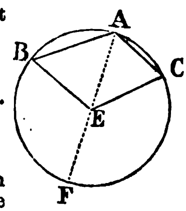
SCH.—I. The reasoning employed anticipates Prop. 5, Book V., and assumes, "that among four magnitudes, if the *first* = *twice* the second, and the *third* = *twice* the fourth, then the *first* + the *third* = *twice* (the second + the *fourth*); and also, that if *one magnitude* = *twice* another, and a *part* from the first = *twice* a part from the second; then the *rem.* of the first = *twice* the rem. of the second." This principle is neither sufficiently self-evident to be received as an axiom, neither has it been demonstrated; *another method of proof for Case III.* has therefore to be adopted.



C.1	Pst. 1 and 2.	Join AE and prod. it to F; and let AB, EC inters. in D.
D.1	32, I.	∵ in △ BED ext. ∠ BDC = ∠ DEB + ∠ EBD;
2	Def.15, I. 5, I.	and ∵ EB = EA, ∴ ∠ EBD = ∠ EAD;
3	Conc.	∴ ∠ BDC = ∠ DEB + ∠ EAD.
4	32, I.	Again, ∵ in △ ADC ext. ∠ BDC = ∠ DAC + ∠ DCA;
5	Def.15, I. 5, I.	and ∵ EA = EC ∴ ∠ DCA = ∠ EAC = ∠ DAC + ∠ DAE;

6	D. 4 and 5.	∴ ∠ B D C = twice (∠ D A C + ∠ D A E);
7	D. 3.	but ∠ B D C = ∠ D E B + ∠ E A D;
8	Ax. 1.	∴ ∠ D E B + ∠ E A D = twice (∠ D A C + ∠ D A E);
9	*Sub.*	take away the com. ∠ E A D;
10	Ax. 3.	∴ rem ∠ D E B = twice rem. ∠ D A C;
		i. e. the ∠ at cen. = twice the ∠ at ⊙ce.

II. If Euclid's definition of an angle (Def. 9, Book I.) be strictly adhered to, the 20th Prop. is geometrically true only when the angle at the centre is less than a rt. angle; but if an angle may be regarded as any angular magnitude less than four right angles, the proposition is universally true. The relation between the angles at the centre and at the circumference, subtended by the same arc, includes the cases in which the angle at the centre is greater than two rt. angles. Take, for instance, the *re-entrant* angle BEC, made up of ∠s BEF, FEC.



C.	Pst. 1 and 2.	Join A E and prod. it to F, the re-entrant ∠ B E C = ∠s B E F, F E C.
D.1	20, III.	∵ ∠ B E F = twice ∠ B A F, and ∠ F E C = twice ∠ C A F;
2	Conc.	∴ re-entrant ∠ B E C = twice ∠ BAC.

In this way the proposition is universally true.

III. The demonstrations often appear *plainer* when arranged exactly as a Simple Equation, and worked by the same rules; thus, in reference to the figures Pr. 20, III.

Case 2.	∠ B E F = 2∠ B A F, or	12 = 6 + 6
and	∠ F E C = 2∠ F A C,	14 = 7 + 7
∴ *Add.*	∠ B E C = 2∠ B A C,	26 = 13 + 13
Case 3.	∠ F E C = 2∠ F A C, or	20 = 10 + 10
and	∠ F E B = 2∠ F A B,	12 = 6 + 6
Sub.	∠ B E C = 2∠ B A C,	8 = 4 + 4

Thus, the sum or difference of the angles at the centre = twice the sum or difference of the angles at the circumference.

Use and App.—1. This proposition is applied in Trigonometry, of which examples will be found in the Use and Application of Prop. 21, Book III. 2. It was employed by Ptolemy to determine the eccentricity of the sun and the epicycle of the moon. 3. In ascertaining the earth's aphelion by three observations the angle at the centre of the orbit is taken double the angle at the circumference.

Prop. 21.—Theor.

The angles in the same segment of a circle are equal to one another.

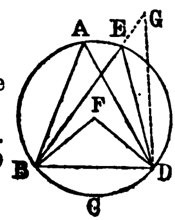
Con. 1, III. Pst. 1 and 2.

Dem. 20, III. Ax. 7. Ax. 2.

E.1	Hyp.	In ⊙ ABCD, let BAD, BED be ∠s in the same seg. BAED;
2	Conc.	then ∠ BAD = ∠ BED.

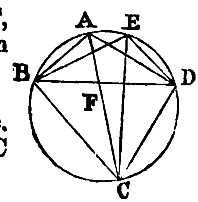
Case I.—*Let the seg.* BAED *be greater than a semicircle.*

C.	1, III. Pst. 1.	Take F the cen. of ABCD and join BF, FD.
D.1	C.	∵ ∠ BFD is at F, the cen. and ∠ BAD at the ⊙ce,
2	"	and ∵ each ∠ has the same base, arc BCD;
3	20, III.	∴ ∠ BFD = twice ∠ BAD.
4	*Sim.*	For the same reason ∠ BFD = twice ∠ BED;
5	Ax. 7.	∴ ∠ BAD = ∠ BED.



Case II.—*Let the seg.* BAED *be not gr. than a semicircle.*

C.	1, III. Pst. 1, 2.	Find F the cen., join AF, and prod. it to C, and join CE, CB, CD.
D.1	C.	∵ AC is a diam. ∴ seg. AEDC is a semic.
2	C.	also segs. BAEDC, DEABC each > a semic.
3	C. and Case I.	and ∵ BAEDC > a semic. ∴ ∠ BAC = ∠ BEC;
4	C. and Case I.	and ∵ DEABC > a semic. ∴ ∠ CAD = ∠ CED;
5	*Add.* Ax. 2.	Hence the whole ∠ BAD = whole ∠ BED.
6	Rec.	∴ *The angles in the same segment, &c.*



Q. E. D.

N.B. This Prop. may be very simply proved; for the angles of the segments are the halves of the same central angle standing on the same common base.

COR.—*If, as in fig. to Case* I, *upon the base*, BD, *of a* △, *there be described a segment*, BAED *of a circle, the vertex of the* △ *shall fall* without, *as at* G, or within, *as at* F, *or* upon the arc, *as at* E, *according as the vertical angle is* <, >, *or* = *the* ∠ A, *in the segment.*

For, if the vertex fall *without*, as at G, then, 16, I., ∠ BED = ∠ BAD and is > ∠ BGD; if *within*, as ∠ F, then, 21, I., ∠ F > ∠ BAD; and if *on*, as E, then 21, III., ∠ BED = ∠ BAD.

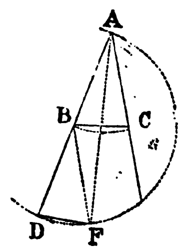
SCH. 1.—Any number of triangles whatever, upon the same base, and with their vertical angles equal, have the *locus* of their vertices in the arc of the same circle; hence, by constructing innumerable triangles fulfilling these two conditions, the vertices would form the segment of a circle.

2. From this Proposition it follows that the angle at the circumference has for its *measure* one half of the arc on which it stands.

USE AND APP.—I. This proposition may be employed for constructing a building in which all the spectators shall see any object under the same angle. A circular amphitheatre is of this kind, for let BD, fig. Case I., be the stage, and BAED the circle in which the seats are placed, then from every one of the seats the stage subtends the same angle of vision; and it is a law in Optics that a line seen under the same angle appears of the same size.

II. *To bisect an angle, as* ∠ DAC.

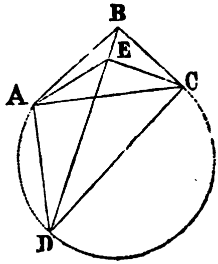
C.1	Pst. 3.	From A with AB desc. an arc BC;
2	3, I. Pst. 3	take BD = BA, and with BD desc. the arc DFA;
3	3, I. Pst. 3.	and from D with rad. = BC cut DFA in F;
4	Pst. 1.	and join BF, AF, BC, and DF,
5	Sol.	then the line AF bisects ∠ A.
D.1	C.	∵ ∠ DBF is at the cen., and ∠ DAF at the ⊙ce;
2	20, III.	∴ ∠ DBF = 2 ∠ DAF;
3	8, I. Ax. 1.	but ∠ BAC = ∠ DBF, ∴ ∠ BAC = 2 ∠ DAF.
4	Conc.	∴ ∠ DAC is bisected by AF.



III. *To construct a figure representative of the distance of the place of observation from an object.*

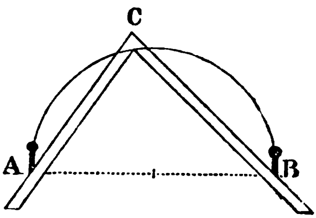
Ex.—Three objects, A, B, C, are distant, AB = 8 miles, BC = 7·2, and AC = 12 miles, from the place of observation D; the ∠ CDB = 25°, and ∠ BDA = 19°; required the distance from D to B.

From a scale of eq. parts, with the given distances, construct the △ ABC; at C make ∠ ACE = 19° = ∠ BDA, and at A, ∠ CAE = 25° = ∠ CDB; and through the three points A, E, and C, (Prop. 3. Use 4, III.) draw a circle ADCE; join BE, and produce it to intersect the circle in D; then BD is representative of the distance from D to B, and is equal to about 15 miles, taken from the scale. For ∠s ACE, ADE, being in the same segment, ADCE, are equal, by 21, III., and also ∠s CAE, CDE, are equal, being in the same segment, EADC. TATES' GEOMETRY, p. 56.



IV. *To draw the arc of any circle*, especially *on a large scale*, by means of two st. pieces of wood fastened so as to form a certain angle, ACB.

Fix into the ground, or on a floor, two pins, A and B, at any required distance, not greater than that of the extremities, A and B, of the angular frame ACB. Move the angular frame round, keeping the sides pressing close to the pins, and a tracing point or pencil at C will mark out the arc of a circle on the ground, or on a floor. The base, AB, remains the same, and the vertices of the triangles have their *locus* in the arc of the same circle.



This method of drawing, or tracing a circle, or the arc of a circle, without having its centre, may be employed for giving a spherical figure to metal cauldrons, or to optical glasses; also for making large Astrolabes, or for marking out the meridian lines, and the lines of latitude on large maps; indeed for every purpose which requires an arc of great magnitude.

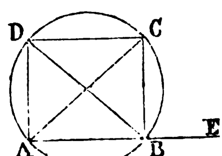
PROP. 22.—THEOR.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

CON. Pst. 1.

DEM. 32, I. 21, III. Ax. 2. Ax. 1.

E.1	Hyp.	Let ABCD be a qu. lat. in the ⊙ ABC.
2	Conc. 1.	then ∠s ABC + ADC = 2 rt. ∠s.
3	,, 2.	& ∠s BAD + BCD = 2 rt. ∠s.
C.	Pst. 1.	Join AC, BD.



CASE I.—*The opp.* $\angle s$ ABC + ADC = 2 *rt.* $\angle s$.

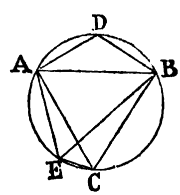
D.1	32, I.	In $\triangle$ CAB, the $\angle$s ABC + BCA + CAB = 2 rt. $\angle$s;
2	21, III.	but $\angle$ CAB = $\angle$ CDB, and $\angle$ ACB = $\angle$ ADB.
3	Ax. 2.	$\therefore$ the whole $\angle$ ADC = $\angle$ CAB, + $\angle$ BCA;
4	*Add.*	to each add $\angle$ ABC, and $\angle$s ADC + ABC = $\angle$s CAB + ABC + BCA;
5	D. 1.	but $\angle$s ABC + BCA + CAB = 2 rt. $\angle$s;
6	Ax. 1.	$\therefore$ $\angle$s ABC + ADC = 2 rt. $\angle$s.

CASE II.—*The other pair of opp.* $\angle s$, BAD, BCD, *also* = 2 *rt.* $\angle s$.

D.1	32, I.	For in $\triangle$ BAD, $\angle$s BAD + ADB + DBA = 2 rt. $\angle$s.
2	D. 2, 4. Case I.	as before $\angle$s BAD + BCD = $\angle$s BAD + ADB + DBA;
3	Conc.	$\therefore$ $\angle$s BAD + BCD = 2 rt. $\angle$s.
4	Rec.	Therefore, *The opposite angles of any quadril., &c.*

Q. E. D.

COR. I.—And conversely, *if the opposite angles of a quadrilateral be together equal to two rt. angles, a circle may be described about the quadrilateral.*



C.1	9, III. Use.	Through A, B, D, the vertices of three $\angle$s, draw a $\odot$ ABD; the $\odot$ce will also pass through the fourth vertex, C.
2	*Ass.* Pst. 1.	For take any other point E in the seg. and join AE, BE.
D.1	22, III., & H.	$\because$ $\angle$ E + $\angle$ D = 2 rt. $\angle$s, and $\angle$ C + $\angle$ D = 2 rt. $\angle$s.
2	*Sub.* Ax. 3.	$\therefore$ taking away com. $\angle$ D, $\angle$ E = $\angle$ C;
3	C.	and these angles have a com. base, AB;
4	Sch. 21, III.	$\therefore$ $\angle$s E and C, are both in the $\odot$ce.
5	Conc.	and $\therefore$ $\odot$ may be described about the qu.lat. ADBC.

COR. II.—*If any side of a quadril. in a circle*, as AB, in fig 1, *be produced* to E, *the ext.* ∠ EBC = *int. opp.* ∠ ADC; for they have a com. supplement ∠ CBA, or each of them together with the int. adj. ∠ = 2 rt. ∠s.

COR. III.—"*If two chords cut off similar segments from the same or different circles, the other segments will also be similar*, since the angles which they contain are supplemental to those in the former segments."

COR. IV.—"*If opp. angles of a quadril. be equal, they must be both rt. angles*, rt. angles being the only equal angles which are supplemental." LARDNER'S EUCLID, p. 110.

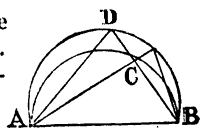
USE AND APP.—PTOLEMY availed himself of this proposition to construct the Tables of Chords; and in Trigonometry it may be applied to prove that the sides of an obtuse angled triangle have the same ratio to one another as the sines of their opposite angles; and hence, if any three be given, the fourth may be found.

PROP. 23.—THEOR.

Upon the same straight line and upon the same side of it there cannot be two similar segments of circles, not coinciding with one another.

CON. Pst. 1, 2.

DEM. 10, III. Def. 11, III. 16, I.



E.	Sup.	If possible, on AB and on the same side of it let there be two sim. segs. ACB, ADB, not coinciding.
D.1	Sup.	∵ ⊙ ACB cuts ⊙ ADB in A and B;
2	10, III.	∴ these ⊙s ACB, ADB cut in no other points;
3	Conc.	∴ one of the segs. falls entirely within the other:

Sup.—*Let seg.* ACB *fall entirely within seg.* ADB;

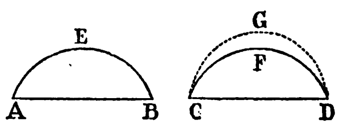
C.1	Pst. 1	In arc ACB take any • C, and join BC
2	Pst. 2	produce BC to D, and join CA, DA.
D.1	H.	∵ seg. ACB is sim. to seg. ADB;
2	Def.11,III.	and sim. segs. of ⊙s contain equal ∠s;
3	Conc.	∴ ∠ ACB = ∠ ADB, *i. e.* the ext. ∠ = the int. ∠;
4	16, I.	but this is impossible;
5	Rec.	Therefore, *there cannot be two similar, &c.* Q. E. D.

Sch.—This proposition, relating to two similar segments of a circle, is the same in principle with the 7th of Book I., which says, that, "on the same base and on the same side of it, there cannot be two triangles which have their sides terminated in one extremity of the base equal, and likewise those terminated in the other extremity equal, not coinciding with one another:" and as the only purpose for which 7, I., was employed was to prove 8, I.; so the purpose to which 23, III. is applied is the demonstration of 24, III. We throw such propositions away as soon as we have used them, yet they are needful links in the chain of geometrical argument.

Prop. 24.—Theor.

Similar segments of circles upon equal straight lines are equal to one another.

Dem. 23, III. Ax. 8. Magnitudes which coincide are equal.



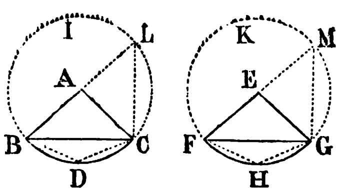
E.1	Hyp.	Let AEB, CFD, be sim. segs. of ⊙s on equal st. lines, AB = CD;
2	Conc.	then seg. AEB = seg. CFD.
D.1	*Super.*	Suppose seg. AEB placed on seg. CFD, so that A be on C, AB on CD, and arcs AEB, CFD, on the same side of CD;

2	H.	then ∵ AB = CD, ∴ B shall coincide with D:
3	D. 2 and 1.	and AB coinciding with CD, and the segs. being on the same side of CD,
4	23, III.	∴ the seg. AEB must coincide with seg. CFD.
5	Sup.	If not, the arc AEB would take another direction, as CGD;
6	Remk.	thus on CD, and on the same side of it, there would be two similar segs. of ⊙s not coinciding, CGD and CFD;
7	23, III.	which is impossible.
8	Conc. Ax.8.	∴ seg. AEB coincides with seg. CFD; and seg. AEB = seg. CFD.
9	Rec.	∴ *Similar segments of circles, &c.* Q. E. D.

Cor. I.—*Similar segments having equal chords have also equal arcs;* to be established by the same principle of superposition.

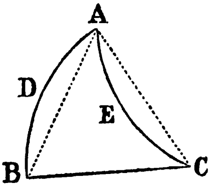
Cor. II.—*Similar segments having equal chords are parts of equal circles*; for circles which agree in more than two points agree in every point.

Cor. III.—*If the radii,* AB, EF, *and angles,* BAC, FEG, *of sectors* BACD, FEGH, *are equal, the sectors themselves are equal.*



C.1	Pst. 1.	Draw the chords BC and FG;
2	Pst. 2, 1,	Produce BA to L, FE to M,
3	Pst. 1.	and join LC, MG, BD, DC, FH, HG.
D.1	4, I.	Then ∵ △ BAC = △ FEG;
2	20, III.	and ∵ ∠ BAC = ∠ FEG, and ∠ BLC = ∠ FMG;

D.3	22, III.	also $\because \angle$ BDC $= \angle$ FHG,
4	Def. 10, III.	$\therefore$ seg. BDC is sim. to seg. FHG.
5	D.1.24, III.	Again $\because$ BC, $=$ FG $\therefore$ seg. BDC $=$ seg. FHG;
6	*Add.*	To each seg. add equal $\triangle$s BAC, FEG;
7	Ax. 2.	$\therefore$ BDC $+$ BAC $=$ FHG $+$ FEG,
		i. e. sect. BACD $=$ sect. FEGH.



Use and App.—*Curve lined figures, as ADB, CEA, are often reduced to rectilineal figures by this proposition.*

For, if two like segments, ADB and AEC of circles are described on the equal sides, AB, AC, of an equilateral $\triangle$ ABC, it is evident that by *transposing* the seg. AEC on ADB, the $\triangle$ ABC $=$ the curve lined fig. ADB, CEA.

Prop. 25.—Prob.

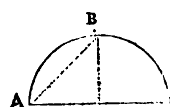
A segment of a circle being given, to describe the circle of which it is the segment.

Con. 10, I. 11, I. Pst. 1. 23, I. Pst. 2.

Dem. 6, I. If a $\triangle$ have two of its $\angle$s equal, then the sides opp. the equal $\angle$s shall be equal.
Ax I. 9, III. Def. 10, I. 4, I.

E.1	Dat.	Let ABC be the seg. of a circle,
2	Quæs.	to desc. the $\odot$ of which it is the seg.
C.1	10, I. 11, I.	Bis. AC by the perp. DB cutting arc ABC in B;
2	Pst. 1.	and join AB;
3	Remk.	there will be *two* cases according as the $\angle$s BAD, ABD are $=$ or $\neq$.

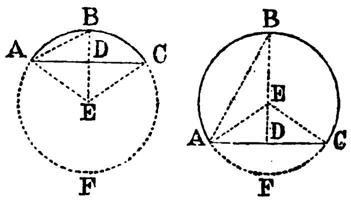
Case I. *Let* $\angle$ BAD $= \angle$ ABD.



D.1	H. 6, I.	$\because \angle$ DBA $= \angle$ DAB,
		$\therefore$ DA $=$ DB;
2	C. & Ax. 1	and $\because$ DC $=$ DA,
		$\therefore$ DA $=$ DB $=$ DC;

D.3	C & D, 2.	And ∵ from D to the ⊙ce, DA, DB, DC are equal;
4	9, III.	∴ D is the cen. of the ⊙ of which ACB is an arc.
5	Sol.	Hence, if from D with rad. DA or DB or DC a ⊙ be desc., it will be that of which ABC in an arc.
6	Remk.	And ∵ the cen. is in AC, ∴ the seg. ABC is a semicircle.

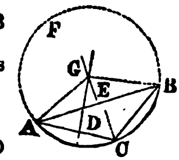
CASE II. *Let* ∠ BAD ≠ ∠ ABD.



C.1	23, I.	At A in AB make ∠ BAE = ∠ ABD;
2	Psts. 2 & 1.	prod., if necessary, BD to meet AE in E, and join EC.
D.1	C. 1. 6, I.	∵ ∠ ABE = ∠ BAE, ∴ BE = AE;
2	C. Def. 10, I.	and ∵ AD = DC, DE com. and ∠ ADE = ∠CDE,
3	4, I.	∴ in △s ADE, CDE, base AE = base EC;
4	D.1, Ax. 1.	but AE = EB, ∴ BE also = EC;
5	D. 3,4.9, III.	and ∵ AE = EB = EC, ∴ E is the cen. of the ⊙.
6	Sol.	Hence, if from E with rad. EA, EB, or EC, a ⊙ be described, it will be that of which ABC is an arc.
7	Remk.	If ∠ ABD > ∠ BAD, the cen. E falls *without* the seg., which therefore is less than a semicircle;
8	Remk.	but, if ∠ ABD < ∠ BAD, the cen. E falls *within* the seg., which therefore is greater than a semicircle.
9	Rec.	∴ *A segment of a circle being given, &c.* Q. E. F.

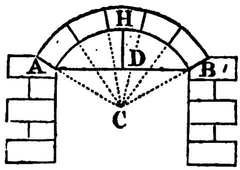
SCH.—This problem might be proposed in another way,—as, *to inscribe a triangle in a circle*; or, *to make a circle pass through three given points, provided they are not in a st. line.* The mode of doing this has been pointed out in Use and App. II., III., and IV., of Prop. 3, Book III., and also in the App.

of Prop. 9, Book III. If, for making a circle pass through three given points, A, B, C, not in a st. line, the method pursued in Prop. 25, Book III., be followed, the process will be—



C.1	Pst. 1.	Join the three given $\cdot$s A, B, C;
2	10, I. 11, I.	bis. A C in D by a perp. from D, and A B in E by a perp. from E;
3	Sol.	the $\cdot$ G where the perpendiculars inters. is the cen. of the required $\odot$.
4	Pst. 1.	Join G A, G C, G B.
D.1	C. 4, I.	$\because$ AD = DC, DG com. and $\angle$s at D equal, $\therefore$ GA = GC.
2	*Sim.*	for a like reason G B = G C,
3	Ax.1.9,III.	$\therefore$ G A = G B = G C; and with either as rad. a $\odot$ may be drawn through A, B, C.

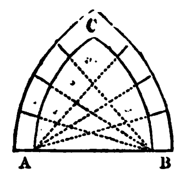
Cor.—In like manner the remainder of the $\odot$ce of a segment of a circle may be completed.



Use and App.—The proposition is of frequent use in all cases when a circle, or an arc, has to be drawn through any three points. Thus:

1. For constructing an arch, of which A B the span and D H the perp. height to the centre, are given:

2. For drawing the plan of a Gothic arch, in which the span A B, and the radii of the arcs intersecting in C are the same:



When A B, the span of the arch, or distance of two of the three points, A, B, is great, the method may be adopted, which is given in Use and App. IV., Prop. 21, Book III., p. 49.

3. For cutting stone, wood, or metal, so that a circle shall pass through three given points:

4. For finding the apogee of the moon, and the eccentricity of the earth's orbit.

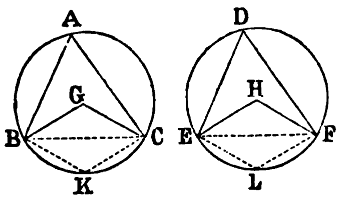
Prop. 26.—Theor.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Con. Pst. 1.

Dem. Def. 1, III. 4, I. Def. 11, III. 24, III. Ax. 3.

E.1	Hyp. 1.	Let ⊙ ABC = ⊙ DEF;
2	„ 2.	and ∠ BGC = ∠ EHF at their centres; and ∠ BAC = ∠ FDE at their ⊙ces;
3	Conc.	then arc BKC = arc ELF.
C.	Pst. 1.	Join BC, and EF.



D.1	H. Def. 1, III.	∵ ⊙ ABC = ⊙ DEF, ∴ BG = GC = EH = HF each to each;
2	H. 4, I.	and ∠ G = ∠ H, ∴ BC = EF.
3	H. Def. 11, III.	And ∵ ∠ at A = ∠ at D, ∴ seg. BAC is sim. to seg. EDF;
4	D. 2. 24, III.	but ∵ BC = EF, ∴ seg. BAC = seg. EDF;
5	H. Ax. 3.	Now ⊙ ABC = ⊙ DEF, ∴ rem. seg. BKC = rem. seg. ELF;
6	Conc.	and arc BKC = arc ELF.
7	Rec.	∴ *In equal circles, equal angles, &c.* Q. E. D.

Cor. 1.—Since by Cor. 4, Prop. 22, bk. III, if the opp. ∠s of a qu. lat. in a circle are equal, those ∠s are rt. ∠s, it follows from Prop. 26, *if the opp. ∠s be equal their opp. diagonal must be a diameter*, and, *the segment a semicircle.*

Cor. 2.—In the same or equal circles one central or circumferential angle is less than, equal to, or greater than another, as the arc of the one is less than, equal to, or greater than, the arc of the other.

Cor. 3.—The diameters which intersect at rt. angles divide the circumference into four equal arcs, or the circle into four equal parts.

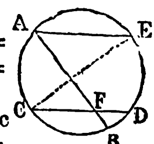
Cor 4.—When the sum of the central angles equals four right angles, the sum of their arcs equals the whole circumference.

Cor. 5.—When the sum of the angles at the circumference equals two rt. angles, the sum of their arcs also equals the whole circumference.

Cor. 6.—Similar arcs of equal circles are equal.

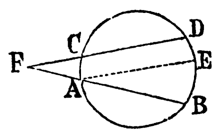
Cor. 7.—*Parallel chords,* AE, CD, *of a circle intercept equal arcs,* AC ED.

Cor. 8.—*If two chords,* AB, CD *intersect* within *a circle,* ABE, *the sum of the intercepted arcs,* AC + DB *is equal to the arc which an angle would intercept at the circumference,* BAE, *that is equal to the angle,* BFD, *under the chords.*



C.	31, I.	Draw AE $\parallel$ CD.
D.1	C.29, I.26, III.	$\because$ AE $\parallel$ CD $\therefore$ $\angle$ EAF = $\angle$ AFC, and $\therefore$ arc ED = arc AC;
2	*Add.*	To arc ED add arc. DB, $\therefore$ arc EB = arc AC + arc DB;
3	D. I.	but $\angle$ BAE *i. e.* FAE = $\angle$ AFC;
4	15, I. Ax. 1.	and $\angle$ AFC = $\angle$ BFD,
5	Conc.	$\therefore$ $\angle$ BAE = $\angle$BFD. Q. E. D.

Cor 9.—*If two chords,* AB, CD, *intersect at a point* F, without *a circle the difference of the arcs,* BD $\sim$ AC, *which they intercept is equal to the arc,* BE, *which an angle* BAE, *would intercept at the circumference that is equal to the angle* BFD, *under the chords.*



C.	31, I.	Draw AE $\parallel$ CD or FD.
D.1	as in Cor. 8.	then arc AC = arc DE
2	*Sub.* Ax. 3.	from arc BD take arc DE, $\therefore$ arc BE = arc BD − arc AC.
3	29, I.	and $\angle$ EAB = $\angle$ BFD.

Lardner's *Euclid p.* 113.

Use and App. 1.—Since, by Cor. 8, of the preceding Prop. $\angle$ BFD = $\angle$ BAE, and by Cor. 20, III, an $\angle$ at the ⊙ce, BAE, on an arc EB is measured by half the arc, BE, on which it stands, namely by $\frac{BE}{2}$ this property

is available *for finding the true central angle of an imperfectly constructed theodolite*, or of any similar instrument, in which the revolving limbs AF, FB, in the last figure but one, are not at the centre of the circle; thus, if the arc AC, as shown by one limb FA, is 50° 20′ and the arc BD, as shown by the other limb FB, is 48° 12′ *the true central angle* $= \frac{50° \, 20' + 48° \, 12'}{2} = \frac{98° \, 32'}{2} = 49° \, 16'$.

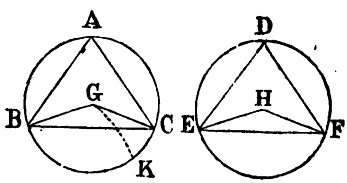
2. We may also apply Cor. 9 to determine the ∠ DFB, when the arcs AC and BD are given; for if arc AC = 48°, and arc BD = 100°, then ∠ DFB = 100° − 48° = 52°.

Prop. 27.—Theor.

In equal circles the angles which stand upon equal arcs are equal to one another, whether they be at the centres or the circumferences.

Con. 23, I.

Dem. 20, III. Ax. 7. 26, III. Ax. 1. Ax. 9.



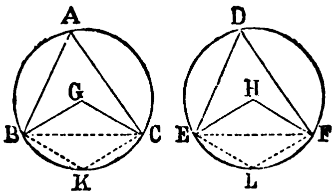
E.1	Hyp. 1.	Let ABC, DEF, be eq. ⊙s, G and H being the centres;
2	Hyp. 2.	and let ∠s BGC, EHF, at G and H, and ∠s BAC, EDF, at the ⊙ces, stand upon the eq. arcs BC, EF;
3	Conc.	then ∠ BGC = ∠ EHF and ∠ BAC = ∠ EDF.

Sup.—If ∠ BGC = ∠ EHF, then (20, III., and Ax. 7) ∠ BAC = ∠ EDF; but if not, one must be the greater.

C.1	Sup.	Let ∠ BGC > ∠ EHF.
2	23, I.	At G in BG make ∠ BGK = ∠ EHF.

D.1	C.26, III.	Then ∵ ∠ BGK = ∠ EHF, ∴ arc BK = arc FE;
2	H.2.Ax.1	but arc EF = arc BC, ∴ arc BK = arc BC;
3	Ax. 9.	*i. e.* the less = the gr., which is impossible;
4	Remk.	∴ ∠ BGC not ≠ ∠ EHF, *i. e.* ∠ BGC = ∠ EHF;
5	20, III.	but ∠ at A = ½ ∠ BGC, and ∠ at D = ½ ∠ EHF;
6	Ax. 7.	∴ ∠ at A = ∠ at D,
7	Rec.	∵ *In equal circles, the angles, &c.* Q. E. D.

Cor. I.—*In the same or in equal circles,* ABC, DEF, *the sectors* BGC, EHF, *which stand upon equal arcs,* BC, EF, *are equal,* and conversely.

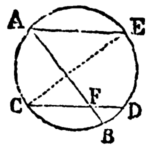


C.	Pst. 1.	Join BC, EF; and from • s K, L, draw KB, KC, LE, LF.
D.1	Def.15,I.27,III.	∵ lines BG, GC = EH, HF, and ∠ BGC = ∠ EHF.
2	4, I.	∴ base BC = base EF, and △ BGC = △ EHF.
3	*Sub.*	From eq. ⊙s ABC, DEF, take equal arcs BC, EF;
4	Ax. 3.	∴ rem. arc BAC = rem. arc EDF;
5	Def. 11, III.	∴ ∠ BKC = ∠ ELF, and seg. BKC is sim. to seg. ELF;
6	D. 2 & 24, III.	but base BC = base EF, ∴ seg. BKC = seg. ELF.
7	*Add.*	To the eq. △s BGC, EHF, add the equal segs. BKC, ELF;
8	Ax. 2.	∴ the sector BGCK = the sector EHFL.

N.B. The *Converse* may be left for the student to demonstrate.

COR. II.—As in Cor. 7, P. 26. III., *if the chords* AE, CD, *of a circle are parallel, they intercept equal arcs,* and vice versâ.

C.	Pst. 1.	Join EC.
D.1	29, I. 26, III.	$\because \angle AEC = \angle ECD$, $\therefore$ arc $ED =$ arc AC.
2	D. 1. 27, III.	Again $\because$ arc $AC =$ arc ED, $\therefore \angle ECD = \angle AEC$;
3	27, I.	$\therefore AE \parallel CD$.



SCH.—1. Propositions 26 and 27 are *converse* propositions; and what is true of equal circles is true of equal arcs in the same circle.

2. As in Cor. 2, Prop. 15, I., all the angles formed by any number of lines diverging from a common centre are together equal to four rt. angles, so the sum of the angles at the centre of a circle subtended by arcs, which together make up the whole circumference, is equal to four rt. angles. Also the sum of the angles at the circumference subtended by those same arcs is equal to two rt. angles.

3. And, since eq. arcs of eq. circles subtend eq. angles, such eq. arcs contain similar segments.

USE AND APP.—1. By Cor. II. of this Prop., a parallel through a given point E to a given st. line CD may readily be drawn; for join EC, and make $\angle CEA = \angle DCE$, and AE is parallel to CD.

2. The principle on which *the area of a sector* is ascertained may be developed from Cor. I. of this proposition, for the area of the triangle BGC, added to the area of the segment BKC gives the area of the sector BGEK.

Or, when the rad. and $\angle BGC$ are given, by principles hereafter to be proved, the Area of the Sector $= \dfrac{\text{Area of} \odot \times \angle BGC}{360°}$,

---

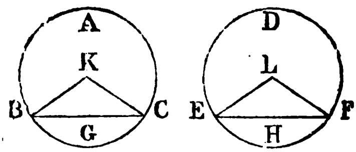
PROP. 28.—THEOR.

In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.

CON. 1, III. Pst. 1.

DEM. Def. 1, III. 8, I. 26, III. Ax. 3

E.1	Hyp. 1.	Let ABC and DEF be eq. ⊙s, and BC, EF eq. st. lines in them;
2	„ 2.	and let BC, EF cut off two gr. arcs BAC, EDF, and two less BGC, EHF;
3	Conc.	then the gr. arc BAC = the gr. EDF; and the less BGC = the less EHF.
C.	1, III. Pst. 1.	Take K, L, centres of the ⊙s, and join KB, KC, LE, LF.



D.1	H. Def. 1, III.	∵ ⊙ ABC = ⊙ DEF, ∴ KB, KC = LE, LF, each to each;
2	H. 8, I.	and $BC = EF$, ∴ $\angle BKC = \angle ELF$;
3	26, III.	∴ the arc BGC = the arc EHF;
4	H. Ax. 3.	but ⊙ ABC = ⊙ DEF; ∴ rem. arc BAC = rem. arc EDF.
5	Rec.	Therefore, *in equal circles, equal st. lines, &c.* Q. E. D.

Sch. As in other instances, the principle of the proposition extends to equal st. lines in the same circle, in which also such equal st. lines cut off equal *arcs.*

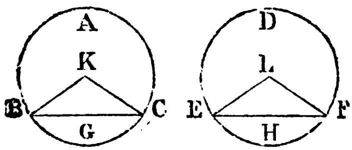
Prop. 29.—Theor.

In equal circles equal arcs are subtended by equal straighl lines.

Con. 1, III. Pst. 1.

Dem. 27, III. Def. 1, III. 4, I.

E.1	Hyp.	Let $\odot$ ABC = $\odot$ DEF, and arc BGC = arc EHF,
2	Conc.	then on joining BC, and EF, chord BC = chord EF,
C.	1, III. Pst. 1.	Find K, L centres of the $\odot$s, and join KB, KC, LE, LF.



D.1	H. 27, III.	$\because$ arc BGC = arc EHF, $\therefore \angle$ BKC = $\angle$ ELF;
3	H.Def.1,III.	and $\because$ $\odot$ ABC = $\odot$ DEF, $\therefore$ BK, KC = EL, LF each to each;
3	D. 1 and 4, I.	and $\angle$ BKC = $\angle$ ELF, $\therefore$ chord BC = chord EF.
4	Rec.	Therefore, *in equal circles equal arcs &c.* Q. E. D.

COR. I.—By the same kind of demonstration it may be shown that, *in the same or in equal circles, equal sectors stand upon equal arcs; and conversely.*

COR. II.—From Prop. 26, 27 and 29, *straight lines which intercept equal arcs are parallel;* and *parallel st. lines intercept equal arcs;* for the alternate angles are equal.

USE AND APP.—1. We may declare generally that whatever has been proved with respect to equal circles is also true when applied to the same circle.

2. In Spherical Trigonometry, Props. 26, 27, 28, and 29 are of continual use. By means of Props. 27 and 28, THEODOSIUS demonstrated that the arcs of the circles of the Italian and Babylonian hours, comprehended between two parallels, are equal; and, in the same way, it may be proved that the arcs of circles of the astronomical hours, comprehended between the two parallels to the equator, are equal.

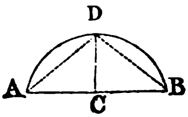
Prop. 30.—Prob.

To bisect a given arc of a circle, i. e., to divide it into two equal parts.

Con. 10, I. 11, I. Pst. 1.

Dem. Def. 10, I. 4, I. 28, III. Cor. 1, III.

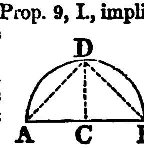
E.1	Datum.	Let ADB be the given arc;
2	Quæs.	it is required to bisect it.
C.1	Pst.1 & 10.11,I.	Join DB, and bis. AB in C by perp. CD;
2	Sol.	then the arc ADB is bis. in D, *i. e.* arc AD = arc BD.
3	Pst. 1.	Join AD, and DB.
D.1	C.1. Def. 10, I.	∵ AC = BC, CD com. and ∠ ACD = ∠ BCD,
2	4, I.	∴ base AD = base BD.
3	28, III.	But eq. st. lines cut off eq. arcs, the gr. = the gr., and the less = the less;
4	Cor. 1, III.	and ∵ DC passes through the cen., arcs AD, DB, each < a semicircle.
5	Conc.	∴ arc AD = arc DB, and ADB is bis. in D.



Q. E. F.

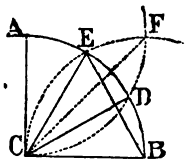
Cor.—Hence, by successive bisections, as in Sch. 1 and 3, Prop. 9, I., *a given arc may be divided into any number of equal parts that are the powers of two*, as 4, 8, 16, 32, &c.

Sch.—1. The bisection of a given rectil. angle, ADB, Prop. 9, I., implies a bisection of ADB, an arc of a circle; but, as by Plane Geometry, a rectil. angle, except in the case of a right angle, DCB, cannot be divided into 3, 5. 6, &c., equal parts, so an arc of a circle, except in the case of a quadrant, DB, which is the circular measure of a right angle, DCB, cannot be cut into 3, 5, 6, &c., equal parts.

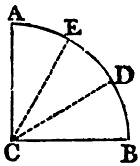


N.B. If, as is mentioned below, other plane curves, besides the circle, had been admttted by Euclid, any angle could also be divided into 3, 5, 6, &c., equal parts.

2. *To trisect a quadrant*, AB, *i. e.* to divide a rt. ∠, ACB, into three equal parts.



C.1	1, I.	On CB const. an equil. △ BEC,
2	9, I.	and bis. its ∠ BCE by CD,
3	Sol.	the rt. ∠ ACB will be trisected, *i. e.* ∠BCD = ∠DCE = ∠ ECA.
D.1	Cor.32,I. C2.	∵ ∠ BCE = $\frac{2}{3}$ of a rt. ∠, and is bis. by CD;
2	Conc.	∴ ∠ BCD = ∠ DCE.
3	H. & D. 1.	Also ∵ ACB is a rt. ∠, and BCE = $\frac{2}{3}$ of a rt. ∠;
4	Ax. 3.	∴ ∠ ACE = $\frac{1}{3}$ of a rt. ∠;
5	Conc.	∴ ∠ ACE = ∠ ECD = ∠ DCB, *i. e.* the quadrant is trisected.

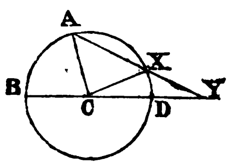


Or,—From A and B, with the rad. of the circle, describe arcs cutting the quadrant in D and E; join EC, DC, and the quadrant is trisected.

3. By successive bisections of the one-third of a rt. ∠, the 1-6th, 1-12th, 1-24th, &c., of a rt. ∠, or of a quadrant is obtained.

4. The division of a quadrant into five equal parts depends on P. 10, IV.

5. Since EUCLID confines himself to straight lines and circles, *the trisection of an angle*, or, *of an arc cannot be effected by his Geometry*; if, however, other curves, formed by the sections of the Cone, were admitted among the curves of our Plane Geometry, the problem could readily be solved. For, "if with two-thirds of any given line, A, as a major axis, an hyperbola be described whose asymptotes," or incoincident lines, "make an angle of 120°; and if with A as a base, and a point on the branch of the hyperbola adjacent to the single third of A as a vertex, a triangle be described, the larger of the angles adjacent to A will always be double of the smaller. Consequently, one of the external angles will be triple of one of its internal and opposite angles; so that by describing on a straight line A a segment of a circle containing the supplement of any given angle less than 180°, that circle will cut the branch of the hyperbola in a point which, being joined with the further extremity of A, will give an angle equal to the given angle."—PENNY CYCL., XXV., p. 260.



6. What is required for the trisection of an arc or of an angle is the solution of the following problem; "from a given point, as A, in the circumference of a ⊙ ABD, to draw a st. line, AXY, such that the part XY, between the circumference and a given diameter BD produced, shall be equal to the radius CA."

D.1	H. 6, I.	∵ CX = XY, ∴ ∠ XCY = ∠ XYC;
2	32, I.	but ∠ AXC or CAX = 2∠ XCD, and ∠ ACB = ∠ XCD + ∠ AXC;
3		∴ ∠ ACB = 3 ∠ XCD, ∴ arc AB = 3 arc XD.
4	Conc.	Thus ∠ XCD = $\frac{1}{3}$ of ∠ ACB, and ∴ arc XD = $\frac{1}{3}$ arc AB.

7. One of the *Trochoidal* curves, known by the name of the *trisectrix*, is peculiarly possessed of the property of dividing any arc into three equal parts.

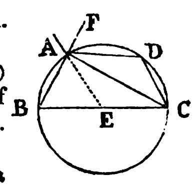
Use and App.—By this problem, the semicircle is divided into quadrants, and the quadrant into arcs of 45°, 22½°, &c.; and the Mariner's Compass, as in Use 4, 9, I., into 32 equal parts called Rhumbs; but the division into single degrees cannot be performed by Euclid's Geometry.

Prop. 31.—Theor.

In a circle, the angle in a semicircle is a right angle, but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Con. 10, I. Psts. 1 & 2.

Dem. Def. 15, I. 5, I. Axs. 1, 2. Def. 10, I. 22, III.
32, I. If a side of a △ be produced, the ext. ∠ = the two int. and opp. ∠s; and the three int. ∠s of every △ are together equal to two rt. ∠s.
17, I. Any two angles of a △ shall together be less than two rt. angles.



E.1	Hyp. 1.	Let ABC be a ⊙, BC its diam. and E its cen.
2	„ 2.	and from C let CA divide the ⊙ into segs. ABC, ADC, of which seg. ABC is > a semic. and ADC < a semic.
3	Conc. 1.	Then ∠ BAC in the semic. is a rt. ∠;
4	„ 2.	∠ ABC in seg. ABC is < a rt. ∠;
5	„ 3.	and ∠ ADC in seg. ADC is > a rt. ∠.
C.1	10, I.	Bisect diam. BC in E; E is the cen.
2	Pst. 1 & 2	join EA, BA; and produce BA to F;
3	Pst. 1.	in arc ADC take any • D, and join AD, DC.

CASE I.—*The* $\angle$ BAC *in the semicircle shall be a rt.* $\angle$.

D.1	Def. 15, I. 5, I.	Now, $\because$ EB = EA = EC; $\therefore$ $\angle$ EBA = $\angle$ EAB, and $\angle$ EAC = $\angle$ ECA;
2	*Add.* Ax. 2.	add the eqs. and the whole $\angle$ BAC = the two $\angle$s ABC, ACB;
3	32, I.	but in $\triangle$ ABC, the ext. $\angle$ FAC = the two int. $\angle$s ABC, ACB;
4	Ax. 1. Def. 10, I.	$\therefore$ $\angle$ BAC = $\angle$ FAC, and $\therefore$ each is a rt. $\angle$.
5	Conc.	$\because$ the $\angle$ BAC in the semic. is a rt. $\angle$.

CASE II.—*The* $\angle$ ABC, *in a seg.* ABC *gr. than a semic., shall be less than a rt.* $\angle$.

D.1	17, I.	And $\because$ in $\triangle$ ABC the $\angle$s ABC, BAC are < 2 rt. $\angle$s.
2	Case I.	and that $\angle$ BAC is a rt. $\angle$;
3	Conc.	$\therefore$ $\angle$ ABC must be < a rt. $\angle$.
4	Rec.	$\because$ the $\angle$, in a seg. > a semic., is less than a rt. $\angle$.

CASE III.—*The* $\angle$ ADC, *in a seg.* ADC < *a semic., shall be gr. than a rt.* $\angle$.

D.1	C. 22, III.	$\because$ ABCD is a quadrilateral in a $\odot$; $\therefore$ $\angle$s ABC, ADC = 2 rt. $\angle$s;
2	Case II.	and $\because$ $\angle$ ABC is < a rt. $\angle$;
3	Conc.	$\therefore$ the other $\angle$ ADC is > a rt. $\angle$.
4	Rec.	$\because$ *In a circle the angle, &c.*

COR.—*If one angle of a triangle be eq. to the other two, it is a rt. angle.*

E.1	Hyp.	For in $\triangle$ ABC, let $\angle$ BAC = $\angle$s ABC + ACB;
2	Conc.	then $\angle$ BAC is a rt. $\angle$.
C.	Pst.	Produce BA to F.
D.1	32, I.	$\because$ $\angle$ FAC = $\angle$s ABC + ACB; $\therefore$ $\angle$ FAC = $\angle$ BAC,
2	Def. 10, I.	$\therefore$ $\angle$ BAC is a rt. $\angle$.

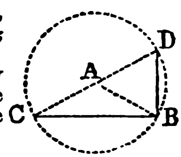
SCH.—The converse of Prop. 31, is, "the segment which contains an acute angle is greater than a semicircle, and that which contains an obtuse angle is less than a semicircle."

2. The Demonstration which LARDNER gives of the 31st Prop. is remarkable for its elegance and brevity; it is founded on Prop. 20, Book III.;—

$\because$ the central $\angle$ on a semicircle $=$ two rt. $\angle$s;
$\therefore$ the circumferential $\angle$ $=$ one rt. $\angle$.
Again, $\because$ the cen. $\angle$ on an arc less than a semic. is less than two rt. $\angle$s;
$\therefore$ the circumferential $\angle$ is $<$ a rt. $\angle$.
And $\because$ the cen. $\angle$ on an arc gr. than a semic. is gr. than two rt. $\angle$s;
$\therefore$ the circumferential $\angle$ is $>$ a rt. $\angle$.

USE AND APP.—I. From the property, Case I., P. 31, III., that the angle in a semicircle is a rt. angle, the following Problems are derived:

PROB. 1. *From a point* B, *in a line, or at the extremity of a line, to draw a perpendicular.*



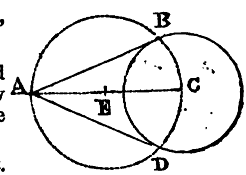
From any $\cdot$ A, out of the line, with the distance A B, describe a semic. meeting C A produced in D, and join BD; then BD is the perp.

N.B. PELITARIUS, a mathematician, often quoted by BILLINGSLEY, gives this Cor. to Pr. 31.—"If in a circle be inscribed a rectangle triangle, the side opposite unto the right angle shall be the diameter of the circle."

PROB. 2.—*From a point* D, without CB *a line, to draw a perp.*

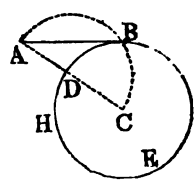
Join D, C, and bisect D C in A; and from A with A C describe a semic. and join AB; DB is the perp. required.

PROB. 3.—*From a point* A, without *a circle, to draw a tangent.*



Join A and the cen. C; bis. AC in E; and from E with E A desc. a circle; and draw A B and A D; then A B and A D are both tangs. from A to the $\odot$ BCD.

The Demonstration may be left to the Student.



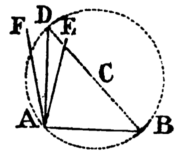
Or, join A, C; 10, I., bis. AC in D, and with DA desc. a semicircle ABC; where the semic. cuts the $\odot$ B, is the tangent point; and AB the tangent from A.

II. By means of a square *the centre of a circle may be easily found;*

For, if B the angular point of the square, CBD (as in the fig. to Prob. 1. above) touch any point in the $\odot$ce; and if also the sides of the square, BD, BC, fall upon two other points, C, D of the $\odot$ce; then the line C D is a diam. and its middle point the centre of the circle.

III. In this proposition workmen possess a *very simple way of trying if their squares be exact;*

For on the hypotenuse D B desc. a semicircle D A B, and apply the angular point A of the square on the ⊙ce of the circle, and one of the sides of the sq. A B, so that the point B of the square may fall upon B the extremity of the diam; then if D, the extremity of the other side of the square, fall upon D the other extremity of the diam. B D, the square is correct; but if that other side, as A E, falls within the circle, the ∠ E A B is < a rt. ∠; if without, as A F, the ∠ F A B is > a rt. ∠.



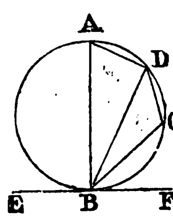
Prop. 32.—Theor.

If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle; the angles which this line makes with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Con. 11, I. Pst. 1.

Dem. 19, III. 31, III. 32, I. Axs. 1, 2 & 3. 22, III.
13, I. The adj. ∠s which one st. line makes with another on the same side of it shall either be two rt. ∠s, or be together equal to two rt. ∠s.

E.1	Hyp. 1.	Let E F touch ⊙ A B C D in B;
2	„ 2.	and from B let B D cut the ⊙;
3	Conc. 1.	then ∠ F B D = ∠ B A D in the altr. seg. D A B.
4	Con. 2.	and ∠ D B E = ∠ B C D in the altr. seg. D C B.
C.1	11, I.	From B draw B A ⊥ E F, cutting the ⊙ in A;
2	Pst. 1.	in arc. D B take any • C, and join A D, D C, C B.



Case I.—*The* ∠ F B D = ∠ B A D *in the altr. seg.* D A B.

D.1	H. and C. 1.	∵ E F touches the ⊙ in B; and B A ⊥ from B;
2	19, III. 31, III.	∴ the cen. of the ⊙ is in B A, and ∠ A D B is a rt. ∠;

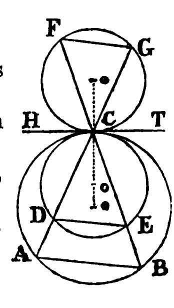
D.3	32, II.	**and ∴ also ∠s BAD, ABD, = a rt. ∠;**
4	C. Ax. 1.	but ABF is a rt. ∠, ∴ ∠ ABF = ∠s BAD, ABD;
5	*Sub.*	take away ∠ ABD;
6	Ax. 3.	∴ rem. ∠ DBF = rem. ∠ BAD, in the altr. seg of the ⊙.

CASE II.—*The* ∠ DBE = ∠ BCD *in the altr. seg.* DCB.

D.1	C.	∵ fig. ABCD is a quadrilateral in a ⊙;
2	22, III.	∴ the opp. ∠s BAD + BCD = 2 rt. ∠s;
3	13, I.	but ∠s DBF + DBE = 2 rt. ∠s;
4	Ax. 1. 5, I.	∴ ∠s DBF + DBE = ∠s BAD + BCD, and ∠ DBF = ∠ BAD;
5	*Sub.*	take away the equals DBF and BAD;
6	Ax. 3.	∴ rem. ∠ DBE = rem. ∠ BCD in the altr. seg. of the ⊙.
7	Rec.	∴ *If a st. line touches a circle, &c.* Q. E. D.

COR. I.—Also, conversely, "*if from the end,* B, *of a line,* DB, *cutting the circle, there be drawn a st. line,* EF, *such that the* ∠*s* FBD, EBD, *which it makes with the cutting line, are equal to the* ∠*s* BAD, BCD, *in the altr. seg. of the* ⊙, *that st. line must touch the circle.*"

COR. II.—*If two or more circles,* ABC, DEC, FGC, *touch each other,* either externally *or* internally, *and through the point of contact,* C, *two st. lines,* AG, BF, *be drawn, meeting their* ⊙*ces, the chords* AB, DE, FG, *of the intercepted arcs will be parallel.*

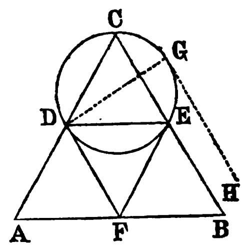


C.	17, III.	Through C, draw HT a tangent.
D.1	11, 12, III.	∵ a line joining the centres passes through C;
2	18, III.	and ∵ HT makes rt. ∠s with that line;
3	16, III.	∴ HT is a tang. to the ⊙s ABC, DEC.
4	32, III.	But ∠ HCF = ∠ CGF in altr. seg. of ⊙ FCG;
5	15, I. Ax. 1	and ∠ HCF = ∠ TCB; ∴ ∠ CGF = ∠ TCB;

D.6	32, III.	also $\angle$ TCB = $\angle$ CAB in the altr. seg. of $\odot$ ABC;
7	Ax. 1.	$\therefore$ $\angle$ CGF = $\angle$ CAB, and they are alt. $\angle$s;
8	27, I.	$\therefore$ FG and AB are parallel.
9	*Sim.*	So it may be shown that FG $\parallel$ DE.
10	Rec.	Therefore, *if two or more circles, &c.* Q. E. D.

Cor. III.—Also, *if two or more circles touch each other, either* internally *or* externally, *at a common* $\cdot$ *of contact,* C, *any line, as* AG, *passing through the* $\cdot$ *of contact, will cut off sim. segments from each; i. e.,* the segs. ACA, DCD, and GCG, are similar.

Cor. IV.—*In an equil* $\triangle$, ABC, *if the sides be bisected in* D, E, F, *and st. lines be drawn joining the points of bisection, of those lines, two* FD, FE, *will be tangents to the circle* DEC, *which passes through* D, E, *the ends of the other line, and* C *the angular point opp. to that line.*



D.1	C.	$\because$ DEF is an eq. lat. $\triangle$;
	Sch.4,32,I.	$\therefore$ $\angle$ FDE $= \frac{2}{3}$ of a rt. $\angle$;
2	C.	also $\because$ ABC is an eq.lat. $\triangle$;
		$\therefore$ $\angle$ ACB $= \frac{2}{3}$ of a rt. $\angle$;
3	Ax. 1.	$\therefore$ $\angle$ C = $\angle$ FDE;
4	C.	but $\angle$ DCE is in the altr. seg. of $\odot$ CDE;
5	Cor.32,III.	$\therefore$ DF, and also EF, touch the $\odot$ in D & E.

Cor. V.—On the same principle, *tangents,* as FD, FE, *through the extremities,* DE, *of the same chord,* DE, *make the angles on the same side* of DE *equal; i. e.* $\angle$ FDE = $\angle$ FED.

Cor. VI.—Also, *if tangents, as* DF, GH, *are parallel, the line* DG *joining the points of contact,* D, G, *is a diameter;* for $\angle$ FDG = $\angle$ HGD, and 29, I., each is a rt. $\angle$; and 19, III., the line DG, from the points of contact, D and G, passes through the centre; $\therefore$ Def. 17, I., DG is a diameter.

Sch.—The 32nd Prop. is sometimes enunciated thus—"If a tangent be drawn to a circle, and from the point of contact a line be drawn cutting off an arc, the angle, between the tangent and the line cutting the circle, will be equal to an angle at the circumference of the circle standing on the arc cut off."

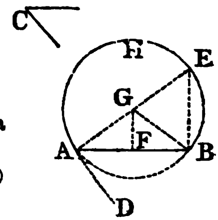
Use and App.—The Proposition is preliminary to the proof of Prop. 33, and is required for the demonstration of all Theorems dependent on the equality of the angles formed by a tangent and secant, and the angles in the alternate segments of the circle.

Prop. 33.—Prob.

Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

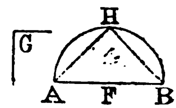
Sol. 10, I. 23, I. 11, I. Pst. 1 & 3.

Dem. 31, III. Def. 10, I. Def. 15, I. 4. I. Cor. 16, III. 32, III. Ax. 1.



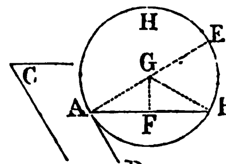
E.1	Data.	Given AB a st. line, and C a rectil. ∠;
2	Quæs.	on AB to desc. a seg. of a ⊙ with an ∠ = ∠ C.

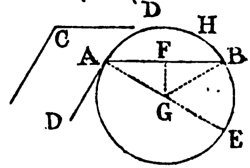
Case I.—*Let the given angle* C *be a rt.* ∠.



C.1	10, I. Pst. 3.	Bis. AB in F; and from F, with FA, desc. the semic. AHB;
2	*Sum.* Pst. 1.	Take a • H in the arc, and join HA, HB;
3	Sol.	then AHB is the seg. required.
D.	C. 31, III.	∵ AHB is a semic.; ∴ ∠ AHB = rt. ∠ C.

Case II.—*But if the given* ∠ C *be not a rt.* ∠.





C.1	23, I.	At A in AB make ∠ BAD = ∠ C;
2	11, I.	from A draw AE ⊥ AD;
3	10.11, I. Pst. 1.	bis. AB in F by perp. FG; and join GB;
4	Sol.	then the seg. AHB is the seg. required.
D.1	C. 3. Def. 10, I.	∵ AF = FB, FG com., and ∠ AFG = ∠ BFG;
2	4, I. Def. 15, I.	∴ AG = GB, and ⊙ from G, with GA, passes through B.
3	*Sum.*	Let this ⊙ be AHB; in seg. AHB the ∠ = ∠ C.

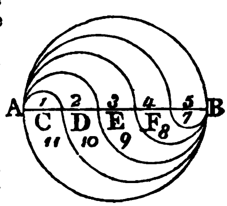
D.4	C.2.Cor.16,III.	∵ at A, AD is $\perp$ to AE a diam., ∴ AD is a tang.;
5	C.	and ∵ AB from the • of contact, A, cuts the ⊙,
6	32, III.	∴ $\angle$ DAB = $\angle$ in the altr. seg. AHB;
7	C. 1. Ax. 1.	but $\angle$ DAB = $\angle$ C; ∴ $\angle$ C = $\angle$ in the seg. AHB.
8	Rec.	∴ *upon the given line is described, &c.*

Q. E. F.

SCH.—Though not belonging to this Proposition, yet from relating to the division of a circle into segments bounded by curved lines, it is *convenient* here to introduce the following Problem:—

To divide a given circle, of which the diameter is AB, *into any number of equal parts, of which the perimeters also are equal.*

By Use and App. 2, P. 34, I., divide the diam. AB into the required number of eq. parts, as C, D, E, F, B; then on one side of the diam., beginning from the extremity A, desc. the semicircles numbered 1, 2, 3, 4, 5, &c., of which the diameters are AC, AD, AE, AF, &c.; and on the other side, beginning from B, the semicircles 7, 8, 9, 10, 11, &c., of which the diameters are BF, BE, BD, BC, &c.; then of the segments bounded by curved lines, 1 + 11 = 5 + 7, 2 + 10 = 4 + 8; and all the parts thus taken in pairs = 3 + 9, and are equal, one curved segment to the other, both in area and perimeter.—LESLIE'S GEOMETRY.



By Def. 1, III., the semicircle, of which AB and BA are diameters, are equal; and ∵ diam. AC = diam. BF, diam. AD = diam. BE, and diam. AE = diam. BD, and diam. AF = diam. BC; ∴ semicircles 1 and 7 are equal, 2 and 8, 3 and 9, 4 and 10, 5 and 11.

If from semic. on AB, equal to semic. on BA, we take semic. on AF and semic. on BC, the remaining space No. 5 = rem. space No. 11; and space 1 = space 7; ∴ the whole area 5 + 7 = the whole area 1 + 11; and in like manner for the areas of all the other curved lined segments.

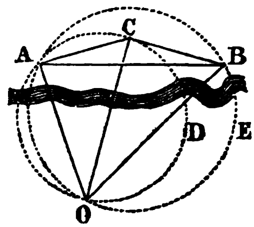
Also the perimeter of semic. on AB = that on BA; of semic. 1 = that of semic 7; of semic. 5 = that of 11, &c.; ∴ the perim. of fig. 5 + 7 = the perim. of 11 + 1; and so on for the perims. of the other curved lined segments.

USE AND APP.—The cases are very numerous in which the 33rd Prop. may be employed.

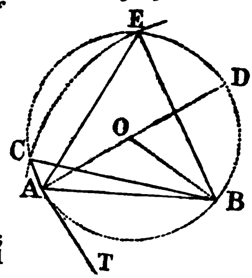
I.—In Coast Surveying, for noting soundings and bearings, this Problem is of great importance and utility; when from a map, or by some other means, the distances of three objects are known, and the observer wishes to ascertain his own position relatively to them. By a sextant the angles at the place of observation between the respective distances are taken; and from these angles and distances the construction or calculation is made. Thus—

Given the distances of three land-marks, A, B, C, *from each other; required their distances from* O, *the place of observation.*

With a sextant take $\angle$s AOC, COB, AOB; through A and B draw a $\odot$ of which the $\angle$ = ACB;
and through A and C draw a circle of which the $\angle$ = AOC;
then, unless the observer be in the $\odot$ce of the $\odot$ passing through A, C, and B, it is evident the $\odot$s ABE and ACD cut in the point O, where the observer stands; $\therefore$ the lines OA, OC, OB, are *representative* of the actual distances from the observer to the three given places.



II. *On a given line* AB, *measuring* 17 *miles to describe a seg. of a circle which shall contain an* $\angle$ *of any fixed number of degrees as* 55°, *being the angle between the line* AB *and* C, *a station* 19 *miles from* B; *and to ascertain the distance of* C *from* A.



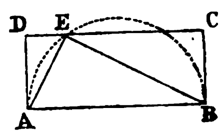
C.1	3, I.	From a scale of eq. pts. set off AB = 17;
2	33, III.	desc. a seg. of a $\odot$ CDB with $\angle$ ACB = 55°;
3	Pst. 3.	from B with rad. = 19 cut the seg. in C and E;
4	Pst. 1	join CA, EA, and EB;
5	Sol.	then AC or AE = the distance required;
6		applied to the scale AC = 3·5 and AE = 19.

N.B. The distance = 19, cuts the seg. in two parts, C and E, and thus the problem has two solutions.

III. *Given the base* AB *and vertical* $\angle$ = $\angle$ TAB, *to find the* locus *of the vertex.*

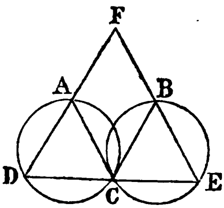
C.	33, III.	On AB desc. a seg. with $\angle$ = $\angle$ TAB.
D.1	21, III.	$\because$ all the $\angle$s in seg. ACDB are equal;
2	Conc.	$\therefore$ the *locus* of the vertex is in that seg.

IV. *Given the vert.* $\angle$, *the base* AB, *and the area to construct a triangle.*



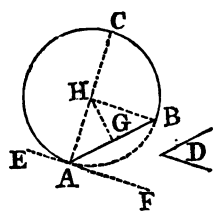
C. 1	(41, I.)	On AB const. rect. ADCB = *twice* the area;
2	33, III.	and on AB a seg. AEB with $\angle$ = the given $\angle$;
3	Pst. 1. Sol.	join EA, EB; and AEB is the req. Δ.
D. 1	41, I.	$\because$ Δ AEB = $\frac{1}{2}$ rect. AC; $\therefore$ Δ AEB is of the area given;
2	C.	and $\angle$ AEB = the vert. $\angle$, and AB is the given base;
3	D. 1, 2.	$\therefore$ Δ AEB is the Δ required.

V. *Through three given points,* A, B, C, *to draw three st. lines, so as to make an equilateral triangle.*



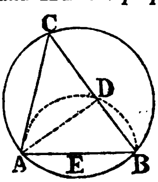
C. 1	33, III.	Join AC, BC, and on each desc. a seg. with an $\angle = \frac{2}{3}$ of a rt. $\angle$, as seg. ADC, BEC;
2	Pst. 1.	through C draw any line ED, cutting the ⊙s in D, E;
3	Psts. 1, 2.	join DA, EB; and prod. them to meet in F;
4	Sol.	then $\triangle$ DEF is the eq. lat. $\triangle$ required.
D. 1	C.	∵ in seg. ADC $\angle$ D $= \frac{2}{3}$ of a rt. $\angle$; and in seg. BEC $\angle$ E $= \frac{2}{3}$ of a rt. $\angle$;
2	Ax. 1.	∴ $\angle$ D $=$ $\angle$ E.
3	32, I. D. 1.	And ∵ $\angle$s D + E + F = 2 rt. $\angle$s; and $\angle$s D + E $= \frac{4}{3}$ rt. $\angle$s;
4	Ax. 3. C.	∴ $\angle$ F $= \frac{2}{3}$ of a rt. $\angle$; and ∴ $\triangle$ DEF is eq. angular;
5	Cor. 6, I.	and ∴ also it is equilateral. Q. E. F.

VI. *Given the* $\angle$ D *equal to the vertical* $\angle$ *of a triangle, and the base* AB, *to find the* locus *of the vertex.*



At A, by 23, I., make the $\angle$ BAF $=$ $\angle$ D; and 11, I., $\angle$ FAH $=$ a rt. $\angle$; bisect, 10 and 11, I., AB in G by the perp. GH; and from H, with rad. HA or HB, desc. a ⊙ ABC; the *locus* of the vertex will be at any point in the arc of the segment ACB.

VII. *Given* AB *the base,* $\angle$ ACD *the vertical* $\angle$, *and* AD *the perp. from the extremity of the base* A *on the opp. side* BC, *to construct the trangle.*



On AB, by 33, III., make a segment containing the given $\angle$ ACD; bisect AB, 10, I., in E; and with rad. EA or EB desc. the semicircle ADB; and from A inflect the perp. AD upon the semicircle in the point D; BDA is a rt. $\angle$, 31, III.; and BD produced to C, and CA joined, give the triangle required.

The Demonstrations may serve as Exercises.

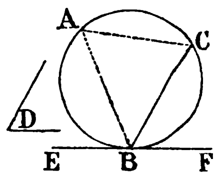
Prop. 34.—Prob.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Con. 17, III. 23, I.

Dem. 32, III. Ax. 1.

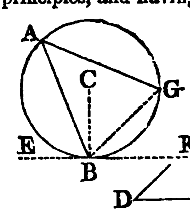
E. 1	Data.	Given the $\odot$ A B C and the rectil. $\angle$ D;
2	Quæs.	to cut from the $\odot$ a seg. with an $\angle$ = D.
C. 1	17, III.	Draw E F a tang. in B to the $\odot$ A B C;
2	23, I.	at B in B F make $\angle$ F B C = $\angle$ D;
3	Sol.	then seg. B A C contains an $\angle$ = $\angle$ D.
D. 1	C. 1.	$\because$ the st. line E F touches the $\odot$ A B C,
2	C. 2.	and $\because$ B C is drawn from B the $\cdot$ of contact;
3	32, III.	$\therefore$ $\angle$ F B C = the $\angle$ in the altr. seg. B A C;
4	C. 2.	but $\angle$ FBC = $\angle$ D; $\therefore$ in the altr. seg. the $\angle$ = $\angle$ D.
5	Conc.	$\therefore$ from the $\odot$ A B C, the seg. B A C is cut off, containing an $\angle$ = the given $\angle$ D. Q. E. F.



Sch.—The following method, on exactly the same principles, and having the same demonstration, gives the *full* construction from Problems 11 and 23, Book I.

From a given $\odot$, A B G *to cut off a segment which shall contain an* $\angle$ *equal to a given rectil.* $\angle$ D.

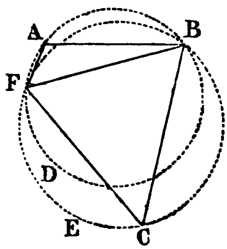
Draw any Rad. C B; and at B, 11, I., draw E F at rt. $\angle$s with C B; at B, the tangent-point of the rad., make the $\angle$ F B G = $\angle$ D; the segment B A G contains an $\angle$ B A G = $\angle$ G B F = $\angle$ D.



Use and App.—I. By the last two Problems, if three observations be taken, the eccentricity of the annual orbit of the earth and its aphelion may be found.

II. Also in Optics, *if two unequal lines*, AB, BC, *are given*, *the point may be ascertained where they will appear equal*, that is, under equal angles.

C. 1	2 & 3, I.	Place AB, BC so as to be conterm. in B;
2	33, III.	on AB, and BC, within the ∠ ABC, construct ⊙s, each containing eq. ∠s in their respective segs. ADB, BCE;
3	Pst. 1.	from F, the · of inters. of the ⊙s, draw FA, FB, FC;
4	Conc.	∠ AFB = ∠ BFC, and the lines AB and BC appear equal from · F.
D. 1	C.	∵ the ∠ in seg. ADB = a certain ∠;
2	C.	and ∵ the ∠ in seg. BCE = the same ∠;
3	Ax. 1.	∴ ∠ AFB = ∠ BFC, and AB appears = to BC. Q. E. F.



Prop. 35.—Theor. *(Very Important.)*

If two st. lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

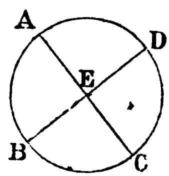
Con. 10, I. 12, I. 1, III. Pst. 1.

Dem. Def. 15, I. 3, III. 47, I. Axs. 1, 2, 3, I.
36, I. Parlms. upon eq. bases and between the same ‖s are equal.
5, II. If a straight line be divided into two equal parts, and also into two uneq. parts, the rect. contained by the uneq. parts, together with the square of the line between the · s of section, is equal to the square of half the line.

E.1	Hyp.	Within ⊙ ABCD let AC and BD cut in E;
2	Conc.	then AE · EC = BE · ED.

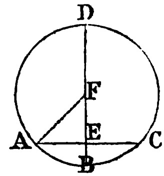
There are *four* Cases of this proposition, according as the intersecting lines pass through the centre of the ⊙ or not.

Case I. — *Let both lines*, AC, BD, *pass through the centre* E.



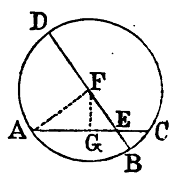
D.1	Def.15, I.	∵ AE = EC = BE = ED;
2	36, I.	∴ AE · EC = BE · ED.

Case II.—*Let* BD, *passing through the cen., cut* AC *not passing through the cen. at rt.* $\angle$*s in the point* E.



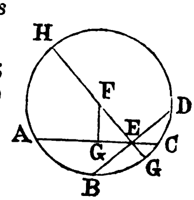
C.	10, I.	Bis. BD in F, the centre; join FA.
D.1	C. & 3, III.	$\because$ BD through F cuts AC not passing through F, and at rt. $\angle$s in E; $\therefore$ AE = EC;
2	C.	and $\because$ BD is div. equally in F, and uneq. in E;
3	5, II.	$\therefore BE \cdot ED + EF^2 = FB^2$, *i. e.* $= FA^2$;
4	47, I.	but $AE^2 + EF^2 = FA^2$;
5	Ax. 1.	$\therefore BE \cdot ED + EF^2 = AE^2 + EF^2$;
6	*Sub.*	take away the com. sq. EF,
7	Ax. 3.	and rem. $BE \cdot ED =$ rem. AE^2; *i. e.* $AE \cdot EC$.

Case III.—*Let* BD *passing through the centre cut* AC, *not passing through the centre, but* not *at rt.* $\angle$*s in the point* E.



C.1	10, I.	Bis. BD as before, & join AF.
2	12, I.	and from F, draw FG, perp. to AC.
D.1	C. 2. 3, III.	$\because$ FG through cen. F cuts AC $\perp$; $\therefore$ AG = GC;
2	C.	and $\because$ AC is divided eq. in G and uneq. in E;
3	5, II.	$\therefore AE \cdot EC + GE^2 = AG^2$ or GC^2;
4	*Add.* Ax. 2.	Add GF^2, $\therefore AE \cdot EC + GE^2 + GF^2 = AG^2 + GF^2$;
5	47, I.	but $GE^2 + GF^2 = EF^2$, & $AG^2 + GF^2 = AF^2$;
6	Ax.1.Def.15, I.	$\therefore AE \cdot EC + EF^2 = AF^2$, *i. e.* $= FB^2$;
7	5, II.	but $FB^2 = BE \cdot ED + EF^2$;
8	Ax. 1.	$\therefore AE \cdot EC + EF^2 = BE \cdot ED + EF^2$;
9	*Sub.* Ax. 3.	take away EF^2; and $AE \cdot EC = BE \cdot ED$.

Case IV.—*Let neither* AC *nor* BD *pass through the centre.*



C.1	1, III. Pst. 1 & 2.	Find the cen. F; join EF; and prod. it to meet the ⊙ in H and G.
D.1	Case III.	$\because AE \cdot EC = GE \cdot EH$; and $BE \cdot ED = GE \cdot EH$;
2	Ax. 1.	$\therefore AE \cdot EC = BE \cdot ED$.
3	Rec.	$\therefore$ *If two st. lines cut one another, &c.* Q. E. D.

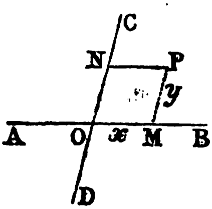
COR.—Conversely, *if the rectangles be equal, contained by two lines intersecting within a circle, the extremities of the lines,* A, C, B, D, *shall be in the circumference of the same circle.*

SCH.—1. The Proposition is sometimes enunciated thus—"If two chords of a circle cut one another, the rectangles under their segments terminating in the point of section shall be equal."

2. The rectangles are each equal to $Rad.^2 - EF^2$.

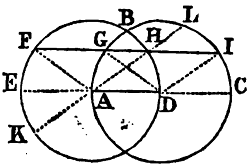
3. When, as in Case I., the fig. B A D is a semicircle, and A E, *the ordinate,* is perp. to B D, *the abscissa,* then the square of the ordinate is equal to the product of the abscissæ, *i. e.* $AE^2 = BE \cdot ED$.

Note.—The terms *ordinate* and *abscissa* may require some explanation When any two st. lines, A B, C D, in the same plane, meet in a common point O, that point may be considered as the *origin* of the lines from which they diverge, and the lines themselves as *axes.* To know the position of any point, as P, in the plane of the axes A B, C D, we must know—1st, between which of the angles P is, whether between ∠s A O C, A O D, B O C, or ∠ B O D; and, 2nd, how far P is from each axis, the distances being measured on parallels to the axes, *i. e.* on the sides of the parallelogram O M P N. Either of the parallels O N or O M is named the *abscissa,* and the other parallels, P N or P M, the *ordinates,* the *abscissa* being the line cut off, the *ordinate* the line which determines the point of section; the *two* are named *co-ordinates,* because together they order or determine the position of the point. Thus, with respect to the point P, if O M be the abscissa, P M is the ordinate. All points in P M have the same abscissa, all in P N the same ordinate. It is usual to denote the length of the abscissa by the letter x, and that of the ordinate by the letter y.



USE AND APP.—I. If of two equal circles, A B C, D B E, the centres A and D be each on the circumference of the other, and a com. chord F G H I be drawn parallel to A D, the line joining the centres, then the lines A F, A H, D G, D I, F G, and H I, joining the points F, G, H, I, where the com. chord cuts the circles, and A, D, the extremities of the line joining the centres, form parallelograms; and if A H be produced to meet the circumferences in K and L, G H = H L and F I = K L.

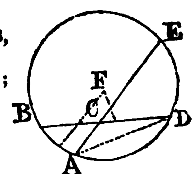
C.1	Pst. 2.	Prod. AD to the ⊙ces in E and C;
2	Pst. 1, 2.	join A H, and prod to K and L;
3	Pst. 1.	join A F, D G, and D I;
4	Conc. 1.	then figs. ADIH and ADGF are ▱s;
5	Conc. 2.	and GH = HL, and FI = KL.



D.1	H.Cor.2.27,III.	$\because$ FI $\parallel$ EC ; $\therefore$ arc FE = arc DH = arc IC ;
2	Cor. 2. 27, III.	and $\therefore$ also $\angle$ IDC = $\angle$ HAD ;
3	28, I. Def. A.	now AH $\parallel$ DI ; $\therefore$ fig. ADHI is a ▱;
4	*Sim.*	and in like manner fig. ADGF is a ▱.
5	34, I.	Hence HI = AD = FG ;
6	35, III.	but GH · HI = AH · HL ; and HI = AH
7	Ax. 3 & 2.	$\therefore$ GH = HL ; and FI = 2 AD + GH ;
8		also 2 AD + GH = KH + HL = KL,
9	Conc.	$\therefore$ FI = KL. Q. E. D.

2. By this Prop. we arrive at a practical way of *finding a line which is the fourth proportional to three given lines,* or *the third proportional to two given lines.*

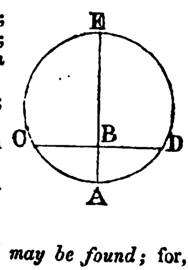
Ex. 1. Let there be *three* lines, AC = 2 eq. pts.. BC = 3, and CD = 4; required a fourth line in proportion to the other three.



C.1	Pst. 2.	From any · B draw an indef. line ;
2	3, I.	on it set the 2nd and 3rd terms, BC = 3, and CD = 4 ;
3	3, I. Pst. 2.	from C set CA = 2 ; and prod. AC indef.;
4	Use 9, III.	through the three · s, A, B, D, desc. a ⊙ cutting AC produced in E ;
5	Sol.	the distance CE is the 4th proportional ; and measured, CE = 6 eq. pts.
D.1	C.	$\because$ within a ⊙ the lines BD and AE cut each other.
2	35, III.	$\therefore$ BC · CD = AC · CE, and CE is the 4th proportional.

If of the four segments any three be given, the fourth may be found; for $\frac{BC \cdot CD}{AC} = CE$; $\frac{BC \cdot CD}{CE} = AC$; $\frac{AC \cdot CE}{BC} = CD$; and $\frac{AC \cdot CE}{CD} = CB$.

Ex. 2. Next, let there be *two* lines, AB = 3 eq. pts. and BC = 6; required a third line in proportion to the other two.



C.1	11, I.	Set the two lines so as to form a rt. $\angle$ at B ;
2	Pst. 2 & 3 I.	prod. AB, CB indef.; and make BD = BC;
3	Use 9, III.	through the three · s, C, A, D desc. a ⊙ cutting AB in E;
4	Sol.	the dist. BE is the 3rd proportional ; and measured, BE = 12 eq. pts.
D.1	C.	$\because$ CD and AE, within a ⊙, cut each other in B ;
2	35, III.	$\therefore$ BC · BD, or BC^2 = AB · BE, and BE is the 3rd proporortional.

If of the three segments any two be given, the third may be found; for, $\sqrt{AB \cdot BE} = BC$; $\frac{BC^2}{AB} = BE$; and $\frac{BC^2}{BE} = BA$.

Prop. 36.—Theor. *(Important.)*

If from any point without *a circle two st. lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.*

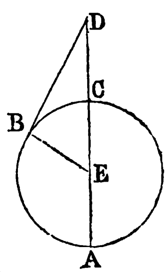
Con. 10, I. 1, III. 12, I.

Dem. 18, III. 47, I. Axs. 1, 2, 3, I. 3, III. 17, III.
6, II. If a st. line be bisd. and produced to any point; the rect. contained by the whole line thus produced, and the pt. of it produced, together with the square of half the line bisd., is equal to the square of the st. line which is made up of the half and the part produced.

E. 1	Hyp. 1	Let D be any · without ⊙ ABC;
2	„ 2	and DCA, DB two st. lines from · D, of which DCA cuts and DB touches the ⊙;
3	Conc.	then $AD \cdot DC = DB^2$

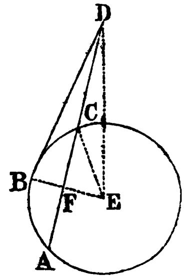
Remk.—Of this Proposition there are two Cases, according as, DCA, the cutting line, passes through the centre of the circle, or not.

Case I.—*Let* DCA *pass through the centre of the* ⊙ ABC.



C. 1	10, I.	Bis. AC in E, and E is the cen. of ⊙ ABC;
2	Pst. 1.	join BE.
D. 1	H. & C. 2	∵ DB is a tang., and BE a line from E to B;
2	18, III.	∴ ∠ EBD is a rt. ∠.
3	C. 1.	And ∵ AC is bisd. in E and prod. to D;
4	6, II.	∴ $AD \cdot DC + EC^2 = ED^2$:
5	Def. 15, I.	but $EC = EB$; ∴ $AD \cdot DC + EB^2 = ED^2$;
6	D. 2 & 47. I.	and EBD being a rt. ∠, ∴ $ED^2 = EB^2 + BD^2$;
7	Ax. 1.	∴ $AD \cdot DC + EB^2 = EB^2, + BD^2$;
8	*Sub.* Ax. 3.	take away EB^2; and $AD \cdot DC = DB^2$, *i.e.* = the square on the tang.

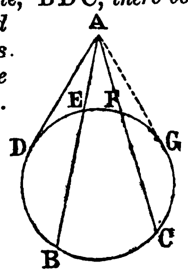
CASE II.—*Let* DCA not *pass through the centre of the* ⊙ ABC.



C.1	1, III. & 12, I.	Find the cen. E, and draw EF perp. to AC;
2	Pst. 1.	and join EB, EC, ED.
D.1	H. & 18, III.	∵ as before, ∠ EBD is a rt. ∠;
2	C. 1.	and ∵ EF, through the cen., cuts at rt. ∠s AC, not through the cen.;
3	3, III.	∴ EF bisects AC, *i. e.* AF = FC:
4	D. 3.	And ∵ AC is bisd. in F, and prod. to D;
5	6, III.	∴ $AD \cdot DC + FC^2 = FD^2$;
6	*Add.*	to each equal add FE^2;
7	Ax. 2.	∴ $AD \cdot DC + FC^2 + FE^2 = FD^2 + FE^2$:
8	D. 1. & 47, I.	but EFD being a rt. ∠, ∴ $ED^2 = DF^2 + FE^2$;
9	D. 1. & 47, I.	and $EC^2 = CF^2 + FE^2$;
10	D. 6. 7. 8. Ax. 1.	∴ $AD \cdot DC + EC^2 = ED^2$;
11	Def. 15. I.	but CE = EB, ∴ $AD \cdot DC + EB^2 = ED^2$;
12	H. & 47, I.	and EBD being a rt. ∠, ∴ $AD \cdot DC + EB^2 = EB^2 + BD^2$;
13	*Sub.* Ax. 3.	take away EB^2, ∴ $AD \cdot DC = BD^2$; *i. e.* = the square on tang.
14	Rec.	∴ *If from any point* without *a circle, &c.*

Q. E. D.

COR I.—*If from any point* A, without *a circle,* BDC, *there be drawn two st. lines,* AB, AC, *cutting it in* E *and* F; *then the rectangles contained by the whole lines and the parts of them without the circle shall be equal to one another; i. e.* BA · AE = CA · AF.

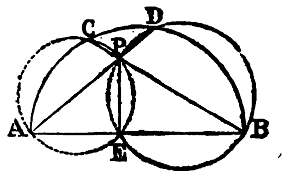


C.1	17, III.	From AD draw a tang. to the ⊙.
D.1	36, III.	∴ $BA \cdot AE = AD^2$, and $CA \cdot AF = AD^2$;
2	Ax. 1.	∴ $BA \cdot AE = CA \cdot AF$.

COR. II.—*If from the same point*, A, *two tangents*, AD, AG, *be drawn to the same circle, they are equal*; for the squares on them are each equal to the same rectangle.

SCH.—Prop. 36 may also be enunciated thus; "If any chord of a circle be produced to cut a tangent to the same circle, the square of the tangent shall be equal to the rectangle under the segments of the chord so produced."

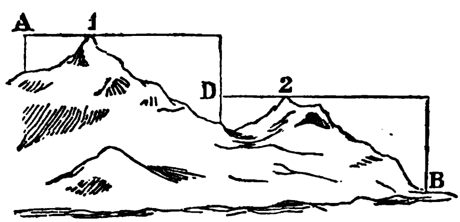
USE AND APP.—I. In any semicircle, ABDC, if from A, B, the extremities of the diameter, chords be drawn, AD, BC, intersecting within the semicircle, the sum of the rectangles formed by any two of such chords, AD, BC, intersecting within it, and by the sections of the chords, AP, BP, between the extremities of the diameter AB and the intersecting point P, shall equal the square on the diameter, AB.



C. 1	12, I.	From P draw PE perp. to AB;
2	Use 9, III.	and draw ⊙s through A, P, E, and B, P, E.
D. 1	Cor. 36, III.	Now in ⊙ APE, $AB \cdot BE = CB \cdot BP$;
2	Cor. 36, III.	and in ⊙ BPE, $BA \cdot AE = DA \cdot AP$;
3	2, II.	Also $AB^2 = AB \cdot BE + BA \cdot AE$;
4	Conc.	$\therefore AB^2 = CB \cdot BP + DA \cdot AP$.

II. From this Prop. is deduced the *Art of taking a true Level* on the earth's surface.

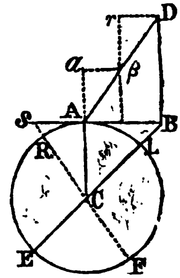
1°. In the more considerable problems of Levelling, it is usual to employ a telescopic sight, and staves with sliding vanes, the distances and observations being entered in a field-book. Thus



BACK SIGHTS.				FORE SIGHTS.		
Station.	Distance.	Height.	Corrected.	Distance.	Height.	Corrected.
1	1420 links	1ft. 5in.	1ft. 4½in.	2448 links.	6ft. 8in.	6ft. 7in.
2	2030 "	6 2	6 1¼	2870 "	7 9	7 7¾
			7 5¾			14 2¾

Then 14ft. 2¾in. *minus* 7ft. 5¾in = 6ft. 9in. the *height* of A above B.

2.° *In measuring an Ascent*, AD, the perpendicular and horizontal lines may be found by means of a spirit level and a staff;



For $a\beta + r\,D = AB$; and $A\,a + \beta r = BD$.

3°. But if the horizontal distance should exceed 100 yards, it will be necessary *to correct it for the curvature;* the horizontal line AB being a tangent, whilst the line of the true level, that of the earth's curvature, AL, is at every point equally distant from the earth's centre. The *deviation* of the horizontal from the true level $= BL$; and, by 36, III., $BE \cdot BL = AB^2, \therefore BL = \frac{AB^2}{BE.}$

N.B. The Greek letters of the diagram a, γ, have been changed to a, r.

Example.—A fountain, B, one mile distant from A, is observed from A, to be on the same horizontal level with the point A; how much is B above A? *i. e.* how much is B farther from the earth's centre C, than the points A or L? CL or CA being 3956 miles.

Here $BC - LC = BL$;
For, by 47, I., $BC = \sqrt{3956^2 + 1^2} = \sqrt{15649937} = 3956{\cdot}0012639$,
Then, $BL = 3956{\cdot}00012639$ *minus* $3956 = {\cdot}00012639$ of a mile $= 8{\cdot}00808$ inches.

4°. By following the same method it will be found that the *deviation* of the horizontal from the true level,—

For 1 mile = 8 inches; 2 miles = 32 inches; 3 miles = 6 feet; and for 4 miles = 10·6 feet, &c.

Or,—Two-thirds of the square of the horizontal level in miles gives the deviation in feet.

5°. The distance within which an object may be seen at sea, or the distance of the *boundary* of the horizon from the spectator is ascertained on the same principle; for the rad. of the horizon $AB = \sqrt{BE \times BL}$.

Ex.—The Peak of Teneriffe, BL, is 2·5 miles above the sea level; what will be the radius of its horizon, or the greatest distance from which it is visible?

Here, $BE = 7912 + 2{\cdot}5 = 7914{\cdot}5$ miles
$\therefore AB = \sqrt{7914{\cdot}5 \times 2{\cdot}5} = \sqrt{19786{\cdot}25} = 140{\cdot}66$ miles.

6°. From an elevation, as the top-mast of a ship, an object will be seen at a greater distance; in this case the *sum of the horizons* of the object and of the elevation will give the whole horizon; *i. e.* $AB + AS = SB$.

Ex.—The top-mast of a ship, RS, is 100 feet above the surface of the sea; at what distance to a spectator standing there will the summit of Teneriffe be visible?

Here $SF = 7912 + {\cdot}01893 = 7912{\cdot}01893$ miles.
$\therefore AS = \sqrt{7912{\cdot}01893 \times {\cdot}01893} = \sqrt{149{\cdot}7745183449} = 12{\cdot}23824$ miles.
As above $AB = 140{\cdot}66$, $\therefore SB = 140{\cdot}66 + 12{\cdot}23824 = 152{\cdot}89824$ miles rad. of the whole horizon.

7°. The *earth's diameter may be ascertained* from knowing the horizon AB, and the height, BL, of an object; for $LE = BE - BL$, and $BE = \frac{AB^2}{BL}$.

Ex.—From the summit of Teneriffe the radius, AB, of the horizon is 140·66 miles, and the height of BL above the sea level is 2·5 miles; required the earth's diameter.

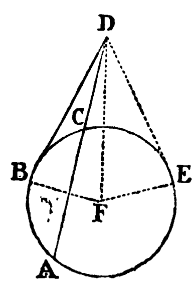
Here $BE = \frac{140·66 \times 140·66}{2·5} = \frac{19785·2356}{2·5} = 7914·098$;
And $LE = 7914·098$ *minus* $2·5 = 7911·598$ miles, earth's diameter.

Prop. 37.—Theor.

If from a point without *a circle there be drawn two lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.*

Con. 17, III. 1, III. Pst. 1.

Dem. 18, III. 36, III. Ax. 1. Def. 15, I. Cor. 16, III.
8, I. If two △s have the three sides of the one respectively equal to the three sides of the other; then these triangles shall be equal in every respect.



E.1	Hyp. 1.	Let D be any · without the ⊙ ABC;
2	„ 2.	and from D two st. lines be drawn, of which DCA cuts the ⊙, and DB meets it, so that $AD \cdot DC = DB^2$;
3	Conc.	then DB shall be a tang. to ⊙ ABC.
C.1	17, III.	Draw DE a tang. in E to ⊙ ABC;
2	1, III.	find F the cen. of the ⊙ ABC;
3	Pst. 1.	and join FB, ED and FE.

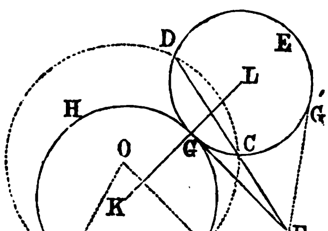
D.1	C. 1 & 18, III.	∵ DE is a tang.; ∴ ∠ FED is a rt ∠ :
2	H. 2 & C. 1.	and ∵ DCA cuts, and DE touches the ⊙ ;
3	36, III.	∴ AD · DC = DE^2;
4	H. & Ax. 1.	but AD · DC = DB^2,
		∴ $DE^2 = DB^2$, and DE = DB.
5	Def. 15, I. D. 4.	Also ∵ FE = FB, FD com. and DE = DB ;
6	8, I.	∴ in △s DEF, DBF, ∠ DEF = ∠ DBF ;
7	D. 1. Ax. 1.	but DEF is a rt. ∠ ; ∴ ∠ DBF is a rt. ∠ ;
8	C.	thus BD is ⊥ to the rad. FB, at its extr. B ;
9	Cor. 16, III.	∴ DB touches the ⊙ at B.
10	Rec.	∴ *If from any point* without *a circle, &c.*

Q. E. D.

Cor.—*Tangents, as* DB, DE, *from the same point,* D, *without a circle are equal.*

D.1	36, III.	∵ DB^2 = AD · DC and DE^2 also = AD · DC ;
2	Ax. 1.	∴ $DB^2 = DE^2$, and DB = DE.

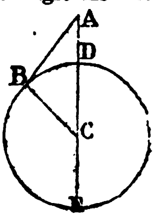
Use and App.—I. By the 36th and 37th Propositions the Problem is solved, "*through two given points,* A, B, *to describe a circle touching a given circle* CDE."



C. 1	*Assum.*	In ⊙ CDE assume any point D of the ⊙ce ;
2	Use 9, III.	and through, A, B, D desc. a ⊙ cutting ⊙ CDE in D, C ;
3	Pst. 1. 2.	join DC, and prod. AB, DC to meet in F ;
4	17, III.	from F draw FG, FG′ tangs. to ⊙ CDE;
5	25, III.	and desc. a ⊙ through ·s A, B, G ;
6	Sol.	then the ⊙ ABH will touch ⊙ CDE in G.

D. 1	C. 4 & 36, III.	∵ FG is tang. to ⊙ CDE ; ∴ $FG^2 = FD \cdot FC$;
2	H.	but A, B, C, D are all in the ⊙ce of ⊙ ABCD;
3	Cor. 36, III.	∴ $FD \cdot FC = FA \cdot FB$.
4	Ax. 1.	∴ also $FA \cdot FB = FG^2$; and ∴ FG is tang. of ⊙ ABH ;
5	Remk.	Now FG is a tang. common to both ⊙s ;
6	Conc.	∴ the ⊙s ABH, CDE touch in the · G.

II. The last three Propositions, 35, 36, and 37, are amongst the most important in Plane Geometry. It was by their aid that MAUROLICO, of Messina, in the sixteenth century, calculated the diameter of the earth; for by the method which has been shown, having ascertained AD the vertical height of a mountain ; the ∠ BAC made by the vertical line, and AB the line from the summit A to B the boundary of sight ; he found the length of AB, by Trigonometry ;



then ∵ $AB^2 = AE \cdot AD$, ∴ $AE = \frac{AB^2}{AD}$; and

$AE - AD = ED$ the diameter of the earth.

REMARKS.

1. In classifying the Propositions of the Third Book, it will be useful to consider them under five general heads :—

1°. The propositions which relate to the Centre of a circle; 1—15 and 20.
2°. To the Tangents of a circle ; 16—19.
3°. To the Segments of circles ; 21—25.
4°. To the Angles in circles, or in their segments ; 26—34.
5°. To the Equality of the rectangles contained by the segments of lines intersecting each other within, or without the circle; 35-37.

2. Of the 37 Propositions which are found in this book only six are Problems, namely :—

Pr. 1. To find the Centre of a circle.
Pr. 17. To draw a Tangent to and from a circle.
Pr. 25. To complete the circle of which a Segment is given.
Pr. 30. To bisect a given Circumference.

Pr. 33. On a given line to draw the segment of a circle containing an angle of a given magnitude; and

Pr. 34. From a circle to cut off a segment which shall contain an angle of a given magnitude.

3. From the Properties of Plane Figures demonstrated in this and in the preceding books, various other problems, however, may be deduced; as

1. To draw a circle through three given points not in a st. line; Use 9, and 25, III.
2. To describe an oval on any given major axis, or a spiral with the radius of the *eye* given; Use 11, III.
3. To join two given points by a serpentine line, or *cima recta*; Use 12, III.
4. To draw a tangent to each of two given circles; Use 18, III.
5. To draw the arc of a chord without knowing the centre of the circle; Use 21, III.
6. To reduce curve-lined figures to rectilineal of equal areas; Use 24, III.
7. Through a given point to draw a line parallel to another line; Use 27, III.
8. To trisect a quadrant; Use 30, III.
9. From a point *in* a line, or *from* the extremity of a line to raise a perpendicular; and from a point *without* a line to drop a perpendicular; Use 31, III.
10. Given the base and vertical $\angle$ of a $\triangle$ to find the *locus* of the vertex; given the vertical angle, the base, and the area of a $\triangle$ to construct the triangle; and given any three points not in a st. line to describe through them an equilateral triangle; Use 33, III.
11. To determine the point where two unequal lines will appear equal, *i. e.* under the same angle; Use 34, III.
12. To find a line proportional to two given lines; and also a line proportional to three given lines; Use 35, III.

4. Were it required in a work like the present, problems might be introduced which show how circles may be drawn which are tangents to two or three given circles, or to two st. lines and a circle; or to two circles and a st. line, &c.; but for these reference may be made to "Geometry, Plane, Solid, and Spherical," Book III., § 8.

5. Several of the principles on which the Levelling and Surveying of Land, and Geographical and Astronomical Observations depend, have been given in the Third Book,—such as the *Methods* :—

1. Of computing the distances and heights of objects when situated on the verge of the natural horizon ; Use 16, III.
2. Of determining the part of a globe which may be enlightened by a luminous body, as by a meteor, volcano, lighthouse, &c. ; of explaining the theory of the phases of the moon ; of ascertaining the distance of the sun ; and of obtaining the dip of the horizon ; Use 19, III.
3. Of constructing a figure representative of the distance of the place of observation from an object ; and of tracing the arc of a circle for giving a spherical figure to optical glasses ; Use 21, III.
4. Of finding the true centre of an imperfectly constructed theodolite, or similar instrument ; Use 26, III.
5. In Coast Surveying, for noting soundings, bearings, &c.; Use 33, III.
6. For taking a true level on the earth's surface ; Use 36, III.; and
7. For calculating the earth's diameter ; Use 37, III.

These are but Examples of the *many useful purposes* to which geometrical science may be applied ; and they may serve to redeem geometry from the prejudiced objection that it is a system of theoretical reasoning without practical results. The practical results are really most important, and in the actual business and occupations of life are an everyday's demand.

GRADATIONS IN EUCLID.

BOOK IV.

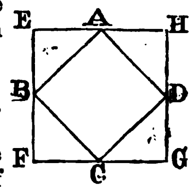
CONTAINING THE METHODS OF CONSTRUCTING REGULAR STRAIGHT-LINED FIGURES IN AND ABOUT A CIRCLE, AND CIRCLES IN AND ABOUT REGULAR STRAIGHT-LINED FIGURES.

Excepting Prop. A, Theor, the fourth book of Euclid's Plane Geometry consists entirely of Problems; it is in fact the Application of the third book to the purposes of inscribing and circumscribing triangles and other regular straight-lined figures in and about circles, or circles in and about such regular figures. The former books supplied the means of drawing regular plane figures of 3, 4, 5, and 15 sides; and by continued bisections of making them of 6, 12, 24, &c., or 8, 16, 32, &c.; or 10, 20, 30, &c., sides. The employment of those means constitutes the object of the book on which we are now entering.

In Trigonometry, Astronomy, and the various departments of Civil and Military Engineering, the fourth book is found of essential service; we also deduce from it the method of obtaining, with sufficient exactness, the quadrature of the circle, and of proving that circles are to one another in the proportion of the squares of their diameters.

Definitions.

1. A rectilineal figure is said to be inscribed in another rectilineal figure when the angular points of the inscribed figure touch the sides of the figure in which it is inscribed, each upon each.



Thus, the fig. ABCD is inscribed in the fig. EFGH.

2. In like manner, a figure is said to be *described* about another figure when the sides of the circumscribed figure touch the angular points of the figure about which it is described, each upon each;

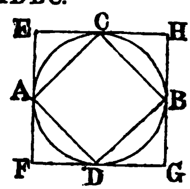
Thus the fig. EFGH is circumscribed about the fig. ABCD.

It is noteworthy that Euclid gives no example of one rectilineal figure being inscribed in another rectilineal figure, or circumscribed about it.

3. A rectilineal figure is said to be *inscribed in a circle* when each angular point of the inscribed figure touches the circumference of the circle;

Thus, the qu. lat. ACBD is inscribed in the circle ADBC.

4. A rectilineal figure is said to be described about a circle when each side of the circumscribed figure touches the circumference of the circle;



Thus, the qu. lat. EFGH is described about the circle ABCD.

5. In like manner, a circle is said to be *inscribed* in a rectilineal figure when the circumference of the circle touches each side of the figure;

Thus, the circle ABCD is inscribed in the quadrilateral EFGH.

6. A circle is said to be *described* about a rectilineal figure when the circumference of a circle touches each corner of the figure about which it is described;

Thus, the circle ABCD is described about the figure ADBC.

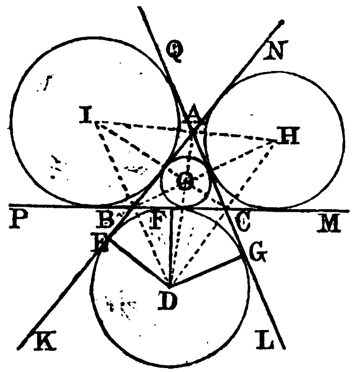
7. A straight line is said to be fitted exactly into a circle, or to be applied in it, when the extremities of it are on the circumference of the circle;

Thus, the lines AC and AD are applied to the circle ABCD.

Definitions Additional to those of Euclid.

8. A circle is said to be *exscribed* to a triangle when, having for centre the point of intersection of any two straight lines that bisect the exterior angles of the triangle, the circle touches a side of that triangle.

Thus, ⊙ EFG from cen. D, (where the lines CD, BD, intersect, which divide the ext. ∠s CBK, BCL, each into two equal parts), and touching in F the side BC of the △ ABC, *is exscribed* to that triangle; as also are the ⊙s from the centres H and I.



9. "Any rectilineal figure, of five sides and angles, is called a pentagon; of six sides and angles, a hexagon; of seven sides and angles, a heptagon; of eight sides and angles, an octagon; of nine sides and angles, a nonagon, of ten sides and angles, a decagon; of twelve sides and angles, a duodecagon; of fifteen sides and angles, a quindecagon," &c.

10. "These figures are included under the general name of *polygons*; and are called *equilateral* when their sides are equal; and *equiangular* when their angles are equal. Also, when both their sides and angles are equal they are called *regular polygons*." Potts' Euclid, p. 124.

N.B.—The force of the propositions in Simpson's Edition is often lessened by not rendering the Greek original into English corresponding, as far as differences of idiom will admit, more closely with Euclid's text. To avoid this, Galbraith and Haughton's rendering of the general enunciation is often followed, though they have not been so thoroughly exact as is desirable.

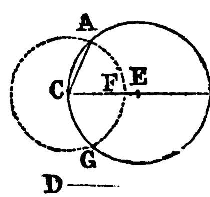
PROPOSITIONS.

Prop. 1.—Prob.

Into a given circle to fit exactly a right line equal to a given right line, which is not greater than the diameter of the circle.

Sol. 1, III. To find the centre of a given circle.
3, I. From the greater of two given lines to cut off a part equal to the less.
Pst. 3 and 1. A circle may be described from any centre at any distance from that centre. A st. line may be drawn from any one point to any other point.

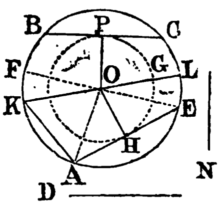
Dem. 15, I. A circle is a plane figure contained by one line, which is called the circumference, and is such that all st. lines drawn from a certain point within the figure to the circumference, are equal to one another.
Ax. 1. Things which are equal to the same thing, are equal to one another.
Def. 7, IV. A st. line is said to be fitted exactly into a circle, or to be applied in it, when the extremities of it are on the circumference of the circle.



E.1	Dat. 1.	Let ABC be the given $\odot$;
2	„ 2.	and D the st. line $\not>$ CB diam. of $\odot$ ABC;
3	Quæs.	in ABC to place a st. line = D.
C.1	1, III.	Find E the cen. of ABC, and draw any diam., BC, through it;
2	Sup.	if BC = D, the required thing is done;
3	H.	but if not, and BC is > D;
4	3, I.	from CB cut off CF = D;
5	Pst. 3 & 1.	and from C, with CF, desc. $\odot$ GFA, and join CA;
6	Sol.	then CA is the line required.

D.1	C.5. Def. 15, I.	∵ C is cen. of ⊙ GFA ∴ CA = CF;
2	C. 4. Ax. 1.	but CF = D, ∴ D = CA.
3	Def. 7, IV.	∴ in ⊙ ABC, a st. line has been placed, CA = the given st. line D. Q. E. F.

USE AND APP.—I. *Within a given* ⊙, ABC, *to place a line of a given length*, D, *not greater than the diam. of the given* ⊙, *which line shall pass through* A, *a given point in the* ⊙*ce.*



C.1	1, IV.	In the ⊙ ABC place BC = D the given line;
2	1, III. 12, I.	find O cen. of ⊙ ABC, and draw OP perp. to BC;
3	Pst. 3.	with OP, from cen. O, draw ⊙ PGH;
4	17, III. Pst. 2.	from A draw AH tang. to ⊙ PGH, and prod. AH to cut ⊙ ABC in E; and join OH;
5	Sol.	then AE = BC = D.
D.1	Def. 15, I. & 14, III.	∵ OH = OP; ∴ AE = BC;
2	C. 1. Ax. 1.	but BC = D; ∴ AE = D. Q. E. F.

II. *To draw that diam. of a* ⊙ *which shall pass at a given distance*, N, *from a given point* A.

C.1	Pst. 3. & 1, IV.	With OA = N, desc. a ⊙ ABC, and in it place AK = N;
2	Sol.	then KO produced to L is the diam. required.
D.1	Def. 16, I. & C.	∵ KL is a diam., and AK = AO = N;
2	Conc.	∴ KL, a diam., passes at the given distance from A.

PROP. 2.—PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

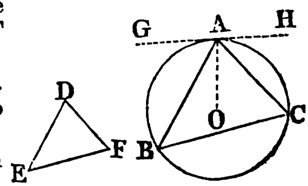
CON. 17, III. To draw a st. line from a given point, either *without* or *in* the circumference, which shall touch a given circle.

23, I. At a given point in a given line to make a rectil. ∠ equal to a given rectil. ∠. Pst. 1.

DEM. 32, III. If a st. line touches a ⊙, and from the point of contact a st. line be drawn cutting the circle, the ∠s which this line makes with the line touching the ⊙, shall be equal to the ∠s which are in the altr. segs. of the ⊙. Ax. 1.

Cor. 3, 32, I. If two △s have two ∠s of the one respectively equal to two ∠s of the other, then the third ∠ of the one shall be equal to the third ∠ of the other.

Def. 3, IV. A rectil. figure is said to be inscribed in a circle when each angular point of the inscribed figure touches the ⊙ce of the circle.

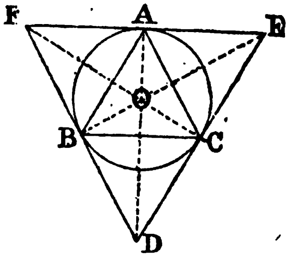


E.1	Data.	Let ABC be the given ⊙, and DEF the given △;
2	Quæs.	in ⊙ ABC to insc. a △ eq. ang. to △ DEF.
C.1	17, III.	To any point, A, in the ⊙ce, draw a tang. GAH;
2	23, I.	at A, in AH, make ∠ HAC = ∠ DEF;
3	23, I.	and at A, in AH, make ∠ GAB = ∠ DFE;
4	Pst. 1.	join BC;
5	Sol.	then △ ABC is the △ required.
D.1	C. 1, 2,	∵ HAG is a tang., and AC from A cuts the ⊙;
2	32, III.	∴ ∠ HAC = ∠ ABC in the alternate seg.;
3	C. 2. Ax. 1.	but ∠ HAC = ∠ DEF; ∴ ∠ ABC = ∠ DEF;
4	*Sim.*	In like manner ∠ ACB = ∠ DFE;
5	Cor. 3. 32, I.	∴ rem. ∠ BAC = rem. ∠ EDF;
6	Def. 3, IV.	Hence △ ABC is eq. ang. with △ EDF, and is inscribed in the ⊙. Q. E. F.

SCH.—The *Analysis* of a problem is a very useful exercise, and, that the learner may become accustomed to the method, some examples will be given. Thus, of Prop. 2, setting out with the admission that the △ ABC has its angles respectively equal to the ∠s D, E, F, the Analysis will be—

Through the point A draw AGH, a tangent to the ⊙;
then, ∵ ∠ CAH = ∠ ABC = ∠ E, ∴ the line AC is given in position; and, being cut by the ⊙ce, the point C is given.
In the same way it will appear that the point B also is given;
and ∵ the three points, A, B, C, are given, ∴ their junction forms △ ABC, inscribed in the circle.

USE AND APP.—*An eq. lat.* △ ABC, *being inscribed in a circle, and through the angular points* A, B, C, *tangents,* DE, EF, FD, *being drawn, these tangents will also form an eq. lat.* △, DEF, *the area of which is four times that of the inscribed eq. lat.* △.



C. 1	1, III.	Find O the cen. of the given circle,
2	Pst. 1.	and join OA, OB, OC, OD, OE, OF.
D. 1	18, III.	∵ ∠s OBD, OCD, are rt. ∠s,
2	47, I.	∴ $OD^2 = OB^2 + BD^2 = OC^2 + CD^2$;
3	Def. 15, I.	but OB = OC, ∴ $BD^2 = CD^2$ and BD = DC;
4	D. 3. and H.	and ∵ OB = OC, BD = DC, and ∠ OBD = ∠ OCD;
5	4, I.	∴ △ OBD = △ OCD, ∠ BOD = ∠ COD, and ∠ BDO = ∠ CDO; *i. e.* ∠s BOC and BDC are each bisected by DO;
6	26, I.	also DO bisects the line BC;
7	3, III.	∴ DO bis. BC at rt. ∠s, and passes through the vertex A.
8	*Sim.*	So, EO bis. AC, and passes to the vertex B; and FO bis. AB, and passes to the vertex C.
9	Cor. 5, I. 32, I.	Now ∠ OBC = $\frac{1}{3}$ of a rt. ∠; ∴ ∠ DBC = $\frac{2}{3}$ of a rt. ∠;
10	*Sim.*	So ∠s DCB, BDC, each = $\frac{2}{3}$ of a rt. ∠,
11	Cor. 6, I.	∴ △ BDC is eq. lat. and = △ ABC;
12	*Sim.*	So △s ACE, ABF are each eq. lat. and = △ ABC;
13	Conc.	∴ △ DEF = 4 △ ABC.
14	D. 11, 12.	Also DE = 2 DC = 2 BC;
15		EF = 2 AE = 2 AC = 2 BC;
16		and FD = 2 FB = 2 AB = 2 BC;
17	Conc.	∴ △ DEF is equilateral. Q. E. F.

Prop. 3.—Prob.

About a given circle to circumscribe a triangle equiangular to a given triangle.

Sol. Pst. 2. 1, III. 23, I. 17, III.

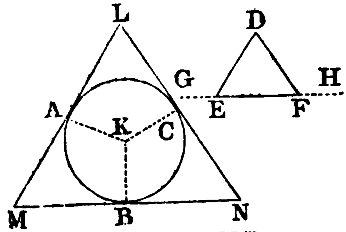
Dem. 18, III. If a st. line touches a ⨀, the st. line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Cor. 1. 32, I. All the interior ∠s of any rectil. fig., together with four rt. ∠s, are equal to twice as many rt. ∠s as the figure has sides.

Ax. 3, I. If equals be taken from equals the remainders are equal.

13, I. The ∠s which one st. line makes with another upon one side of it are either rt. ∠s, or together equal to two rt. ∠s. Ax. 1.

Def. 4, IV. A rectil. fig. is said to be described about a ⨀, when each side of the circumscribed fig. touches the ⨀ce of the ⨀.



E.1	Data.	Let ABC be the given ⊙, & DEF the given △;
2	Quæs.	about the ⊙ ABC to desc. a △ eq. ang. to the △ DEF.
C.1	Pst. 2.	Produce EF both ways to G, H;
2	1, III. Pst. 1	find K the cen. of ⊙ ABC, and draw KB;
3	23, I.	at K in BK make ∠ BKA = ∠ DEG, and ∠ BKC = ∠ DFH;
4	17, III.	and through A, B, C, draw LM, MN, NL, tangs. to ⊙ ABC;
5	Sol.	then △ LMN shall be the △ required.
D.1	C. 4.	∵ LM, MN, NL, are tangs to ⊙ ABC;
2	C. 2, 3.	and KA, KB, KC, lines from the cen. to A, B, C;
3	18, III.	∴ the ∠s at A, B, and C are all rt. ∠s.
4	Cor. 1, 32, I.	And ∵ the 4 ∠s of the qu. lat. AMBK = 4 rt. ∠s;
5	D 2.	and two of the four, KAM, KBM, are rt. ∠s;
6	Ax. 3.	∴ the other two ∠s, AKB, AMB = 2 rt. ∠s;
7	13, I.	but ∠s DEG, DEF = 2 rt. ∠s;
8	Ax. 1.	∴ ∠s AKB, AMB, = ∠s DEG, DEF;
9	C. 3. Ax. 3.	and ∠ AKB = ∠ DEG, ∴ ∠ AMB = ∠ DEF.
10	*Sim.*	In like manner ∠ LNM = ∠ DFE;
11	32, I. Ax. 3.	∴ rem. ∠ MLN = rem. ∠ EDF;
12	Def. 4. IV.	∴ △ LMN is eq. ang. with △ DEF; and is described about the ⊙ ABC. Q. E. F.

SCH.—*Analysis:* We suppose the problem to have been solved, the △ LMN being described about the given ⊙ ABC, so that ∠ L = ∠ D, ∠ M = ∠ E, and ∠ N = ∠ F.

Join K the cen. of the ⊙ to the tang. points A, B, C.
In the qu. lat. BKCN, the four ∠s = four rt. ∠s;
and ∵ ∠s KBN, KCN, are 2 rt. ∠s; ∴ ∠s BKC, BNC = 2 rt. ∠s.
But ∠ N being given, its supplement ∠ BKC is also given;
consequently, KB and KC, the two radii, are given in position.
Thus, it may be shown that the ∠ AKB is given, and the line KA given in position.

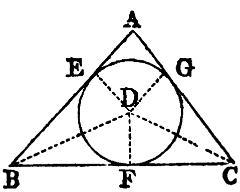
The inters. of KA, KB, KC, with the ⊙ce, or the points A, B, C, are given;
∴ the tangs. MN, NL, LM, are given in position.
Thus the △ LMN is given.

PROP. 4.—PROB.

To inscribe a circle in a given triangle.

CON. 9, I. To bisect a given rectil. $\angle$, *i.e.* to divide it into two equal parts
12, I. To draw a perp. to a given st. line of unlimited length from a given point without it. Pst. 3.

DEM. Ax. 11. All rt. $\angle$s are equal to one another.
26, I. If two Δs have two $\angle$s of the one equal to two $\angle$s of the other, each to each, and one side equal to one side; then shall the other sides be equal, each to each, and also the third $\angle$ of the one to the third $\angle$ of the other. Ax. 1.
9, III. If a point be taken within a circle, from which there fall more than two equal st. lines to the $\odot$ce, that point is the cen. of the $\odot$.
Cor. 1, 16, III. If a st. line be drawn at rt. $\angle$s to any diam. of a $\odot$; from its extremity, it shall touch the $\odot$ at the extremity; and a st. line touching the $\odot$ at one point shall touch it at no other point.
Def. 5, IV. A $\odot$ is said to be inscribed in a rectil. fig. when the $\odot$ce of the $\odot$ touches each side of the figure.



E.1	Datum.	Let ABC be the given Δ;
2	Quæs.	in it to inscribe a $\odot$.
C.1	9, I.	Bis. $\angle$s ABC, ACB, by BD, CD, meeting in D;
2	12, I.	from D draw DE, DF, DG, perps. to AB, BC, CA.
3	Pst. 3.	with DF as rad. draw a $\odot$ EFG;
4	Sol.	then the $\odot$ EFG is the $\odot$ required.
D.1	C. 1, 2.	$\because \angle EBD = \angle FBD$, $\angle DEF = \angle DFB$, and BD com.;
2	26, I.	$\therefore DE = DF$;
3	*Sim.* Ax. 1.	So $DG = DF$; $\therefore DE = DG = DF$;
4	9, III.	$\therefore$ the points E, F, G, are in the $\odot$ce of the $\odot$;
5	C. 2.	And $\because$ the $\angle$s at E, F, G, are rt. $\angle$s;
6	Cor. 1. 16, III.	$\therefore$ AB, BC, CA touch the $\odot$ EFG,
7	Def. 5. IV.	and $\therefore$ $\odot$ EFG is inscr. in Δ ABC.

Q. E. F.

SCH.—1. In other words, the 4th Prop. is—*To describe a $\odot$ which sha touch three given st. lines not parallels.*

2. In the *Analysis* of this problem we assume that the $\odot$ EF G has been inscribed in the Δ ABC.

The cen. of the ⊙ being D, join DA, DB, DC.

Now ∵ DE = DF, DB, com. to △s BFD, BED, and the ∠s at E and F are equal. being rt. ∠s;

∴ △ BFD = △ BED, ∠ DBF = ∠ DBE, and thus BD is given in position.

By a similar argument it may be shown that DA and DC are given in position :—

∴ their point of inters., D, is also given.

Use and App.—I. Connected with this problem is the following theorem : *If three ∠s,* A, B, C, *of a △* ABC *be bisected by st. lines,* BD, CD, AD, *these lines will intersect in the same point.*

C.	12, I.	From D, the inters. of BD and CD, draw DE, DF, DG, perps. to AB, BC, and CA.
D.1	C. and H.	∵ ∠s DFC, DCF = ∠s DGC, DCG, and DC is com.;
2	26, I.	∴ FD = GD.
3	*Sim.* Ax. 1.	So ED = GD; ∴ ED = FD;
4	Conc.	∴ the lines BD, CD, AD, have a com. point of inters. D.

II. An expression for the *Area of a Triangle,* and for *the Radius of the inscribed circle* may be deduced from this theorem.

First.—For the *Area of the Triangle,* the sides AB $= a$, BC $= b$, and CA $= c$, and the radius, DF $= r$, of the inscribed ⊙ being given;

The △ ABC = △s ABD + BDC + CDA;

the area of ABD $= a \times \frac{r}{2}$; of BDC $= b \times \frac{r}{2}$ and of CDA $= c \times \frac{r}{2}$;

∴ the area of △ ABC $= (a + b + c)\,\frac{r}{2}$.

Second.—For the *Radius of the inscribed circle,* the area and sides being given;

$$r = \frac{2 \text{ area of } \triangle}{a + b + c}.$$

Ex. 1. The sides of a △ are 52, 56, and 60 yards, and the rad. of the inscribed ⊙ 16 yards, required the area.

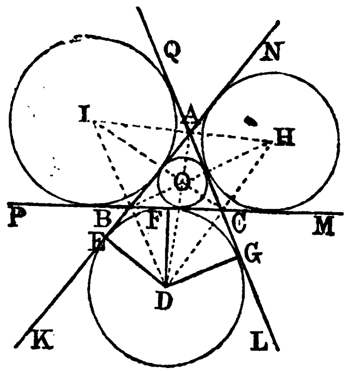
Here $(52 + 56 + 60) \times \frac{16}{2} = 168 \times 8 = 1{,}344$ square yards.

Ex. 2. The area of a △ is 216 square feet, and the sides 18, 24, and 30 feet, required the radius of the inscribed circle.

Here, $\frac{2 \times 216}{18 + 24 + 30} = \frac{432}{72} = 6$ feet.

III. The Corollaries to Prop. 4, given by Galbraith and Haughton, in their "Manual of Euclid," Bk. IV., pp. 6, 7, open up an interesting subject of inquiry, namely—*The Properties of Circles* exscribed *to a Triangle.* From those Corollaries we select the following, referring to Def. 8, IV. :—

1. The bisectors of any internal $\angle$, as D A of $\angle$ B A C, and of the remaining two ext. $\angle$s, as D B and D C of $\angle$s C B K, B C L, intersect in the same point, D, which is the cen. of the $\odot$ E F G, touching the side, BC, opposite the given $\angle$ B A C and the productions, B K, C L, of the two sides A B, A C, containing it.



2. The radius, as D F, of the exscribed $\odot$, E F G, may be found in numbers, by dividing the area of the $\triangle$ A B C, by half the difference between the sum of the two sides and the base ; because the whole triangle is the difference between the sum of the two triangles ACD and ABD, whose bases are the sides AB, AC, and common altitude the radius of the exscribed circle, and the $\triangle$ B C D, whose base is the third side B C, and altitude the same radius D F.

3. From 13, II., a Formula is deduced for calculating the *Area of a Triangle* in the terms of the sides ; $2s$ being the perimeter or sum of the sides, and s the semi-perimeter, and a, b, c, the sides of the triangle :—

$$\text{Area of } \triangle = \sqrt{s\ (s-a)\ (s-b)\ (s-c)}.$$

4. By aid of this Formula and Cor. 2, given above, we can readily express *the radii of the three exscribed circles;* r denoting the radius of the inscribed circle, and r', r'', r''', the radii of the circles exscribed to the sides opposite A, B, C, respectively ;

$$\text{Then, } r' = \sqrt{\frac{s \cdot s-b \cdot s-c}{s-a}},\quad r'' = \sqrt{\frac{s \cdot s-a \cdot s-c}{s-b}},\quad r''' = \sqrt{\frac{s \cdot s-a \cdot s-b}{s-c}};$$

$$\text{and by Use 2, P. 4, IV.}\quad r = \frac{2\ area}{a+b+c},\ \text{or}\ r = \sqrt{\frac{s-a \cdot s-b \cdot s-c}{s}}.$$

5. By taking the product of the four radii, $r\,r'\,r''\,r'''$, the following remark able expression is obtained for the area of a triangle :—

$$\text{Area of } \triangle = \sqrt{r\,r'\,r''\,r'''}.$$

Ex. 1. Find the radii of the inscribed and exscribed circles in the triangle of which the sides are $a = 100$, $b = 86$, and $c = 72$.

$$\text{Here } s = \frac{100+86+72}{2} = \frac{258}{2} = 129,\ s-a = 29,\ s-b = 43,\ \text{and } s-c = 57.$$

$$\text{And}\quad r = \frac{2\ Area}{a+b+c}\ \text{or}\ r = \sqrt{\frac{s-a \cdot s-b \cdot s-c}{s}}.$$

Now area $= \sqrt{129 \times 29 \times 43 \times 57} = \sqrt{9169191} = 3028 \cdot 06$; and

$$r = \frac{6056 \cdot 12}{258} = 23 \cdot 473, \text{ or } r = \sqrt{\frac{29 \times 43 \times 57}{129}} = \sqrt{551} = 23 \cdot 473$$

$$r' = \sqrt{\frac{129 \times 43 \times 57}{29}} = \sqrt{\frac{316179}{29}} = \sqrt{10902 \cdot 7} = 104 \cdot 416$$

$$r'' = \sqrt{\frac{129 \times 29 \times 57}{43}} = \sqrt{\frac{213237}{43}} = \sqrt{4959} = 70 \cdot 420$$

$$r''' = \sqrt{\frac{129 \times 29 \times 43}{57}} = \sqrt{\frac{160863}{57}} = \sqrt{2822 \cdot 1579} = 53 \cdot 124$$

Ex. 2. From the two formulas given above, find the area of the foregoing triangle :—

1°. Area $= \sqrt{129 \times 29 \times 43 \times 57} = \sqrt{9169191} = 3028 \cdot 06$.

2°. Area $= \sqrt{23 \cdot 473 \times 104 \cdot 416 \times 70 \cdot 420 \times 53 \cdot 124} = \sqrt{9168919 \cdot 3231896}$ $= 3028 \cdot 0223$.

6. In a rt. angled △ the *diam.* of the *inscribed* ⊙ is equal to the difference between the sum of the sides and the hypotenuse; and the *diam.* of the ⊙ *exscribed* to the hypotenuse is equal to the perimeter of the triangle.

N.B. For other properties of the inscribed, circumscribed, and exscribed circles of a given triangle, consult the Appendix to Galbraith and Haughton's *Manual of Euclid*, Bk. IV., p.p. 20-25.

Prop. 5.—Prob.

To circumscribe a circle about a given triangle.

Con. 10, I. To bisect a given finite st. line. Pst. 1 and 3.

11, I. To draw a st. line at rt. ∠s to a given st. line from a given point in the same.

Dem. Ax. 9. The whole is greater than its part.

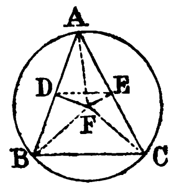
Ax. 12. If a st. line meets two st. lines, so as to make the two int. ∠s on the same side of it taken together less than two rt. ∠s, these two st. lines, being continually produced, shall at length meet upon that side on which the ∠s are less than two rt. ∠s.

Def. 10, I. When a st. line standing on another st. line makes the adj. ∠s equal to each other, each of these ∠s is called a *rt.* ∠; and the st. line which stands on the other is called a *perp.* to it.

4, I. If two △s have two sides and the incl. ∠ of one equal to two sides and the incl. ∠ of the other, the △s are equal in every respect. Ax. 1.

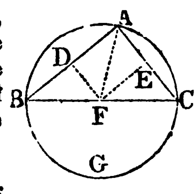
Def. 6, IV. A circle is said to be *desc.* about a rectil. fig. when the ⊙ce of the ⊙ touches each corner of the fig. about which it is described.

31, III. In a ⊙ the ∠ in a semi ⊙ is a rt. ∠; but the ∠ in a seg > a semi ⊙ is less than a rt. ∠; and the ∠ in a seg. < semi. ⊙, is greater than a rt. ∠.



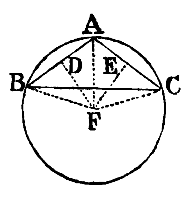
E. 1	Datum.	Let A B C be the given △;
2	Quæs.	To desc. a ⊙ about the △ A B C.
C. 1	10, I.	Bisect A B, A C in D, E;
2	11, I. Post. 1.	and from D, E, draw D F, E F ⊥s to A B, A C and join D E;
3	C. 2.	then ∵ ∠s A D F, A E F are rt. ∠s,
4	Ax. 9	∴ ∠s E D F, D E F are < two rt. ∠s;
5	Ax. 12.	and ∴ D F, E F being prod. shall meet;
6	Sup. Pst. 1.	let them meet in F, and join F A;
7	Sup.	If F be not on B C, join also F B and F C;
8	Pst. 3.	with F A or F B desc. a ⊙;
9	Sol.	the ⊙ A B C is c. scr. about the △ A B C.
D. 1	C.1.2. Def.10, I	∵ A D = D B, D F com., and ∠ A D F = ∠ B D F;
2	4, I.	∴ △ A D F = △ B D F, and A F = F B.
3	*Sim.* Ax. 1.	So C F = F A, ∴ B F = F C;
4	D. 2, 3.	thus F A = F B = F C;
5	Rem.	∴ a ⊙ from F, with rad. F A or F B, or F C, will pass through the extrs. of the other two;
6	Def. 6, IV.	∵ the ⊙ passes through the angr. · s A, B, C;
7	Sol.	and ∴ the △ A B C is c. sc. by the ⊙ A B C.

Q. E. F.



Cor. 1. When F, the centre of the ⊙, falls *within* the △, each angle is an acute ∠, (31, III.); when *on* a side of the △, the ∠ opp. that side is a rt. ∠, (31, III.); and when *without* the △, the ∠ opp. the side nearest the centre is an obt. ∠, (31, III.); and *conversely*.

If the given △ be acute angled, the cen. of the ⊙ falls *within* it; if rt. angled, the centre is *on* the side opp. the rt. ∠; and if obtuse angled, the centre falls *without* the triangle, beyond the side opp. the obtuse angle.



2. The perps. bisecting the sides of a △ meet at the centre of the circumscribing circle.

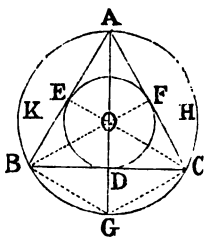
3. Perpendiculars from each ∠ on the opp. side intersect in the same point.

Sch. 1. This proposition is identical; 1° with that (in 3, III. and 9, III.) of describing a circle through three given points, A, B, C, not in the same st. line; 2° through two given points, A, B, and touching a given st. line; or 3° through a given point A, and touching two given st. lines.

2. It is also identical with describing a circle; 1° through two given points and touching a given circle; 2° through a given point and touching two given circles; or 3° touching three given circles.

3. And it may be further extended to solve the problems of describing a circle; 1° through a given point, touching a given st. line, and also touching a given circle; 2° touching two given st. lines and also a given circle; or 3° touching a given st. line, and two given circles. See *Geom. Plane, Solid, and Spherical*, pp. 114, 118.

Use and App.—"*If one circle* EDF *is inscribed in an equil.* △ ABC *and another* KAH *circumscribed about it, the circles* EDF, KAH *are* concentric, *and the diam* OG *of one is* double *the diam.* OD, *of the other.*

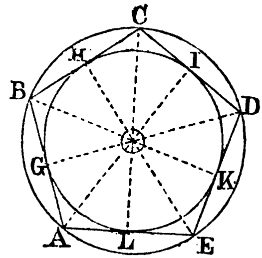


C. 1	9, I.	Bis. the ∠s A, B, C by the st. lines AG, BF, CE;
2	Pst. 1.	and join BG, GC.
D. 1	H. and C. 1.	∵ BA = CA, AG com., and ∠ BAD = ∠ CAD;
2	4, I. Def. 10, I.	∴ BD = DC, and ∠s at D rt. ∠s.
3	*Sim.*	So AE = EB, and AF = FC, and ∠s at E, F, are rt. ∠s.
4	Cor. 1, III.	∴ AG, CE, BF, pass through O the cen. of ⊙ ABC;
5	1, III.	and O, the cen. of AG a diam., is the cen. of ⊙ ABC.
6	H. and C.	Again ∵ EB = BD, OB com., and ∠ EBO = ∠ DBO;
7	4, I	∴ OE = OD;
8	*Sim.* Ax. 1.	So OF = OD; ∴ OE = OF = OD;
9	Cor. 1, III.	∴ O is also the cen. of the ⊙ EDF.
10	Cor. 31. III.	Next, ∵ AKG and AHG are sem. cs.;
11	Ax. 11,	∴ ∠ ABG = ∠ ACG;
12	H.	and ∵ △ ABC is eq. ang.; ∴ each ∠ = $\frac{2}{3}$ of a rt. ∠;
13		∴ also ∠ DBG = $\frac{1}{3}$ rt. ∠ = ∠ DCG.
14	D. 2.	Now ∠ BDO = ∠ CDG, ∠ DBO = ∠ DCG, and BD = DC:
15	26, I.	∴ OD = DG, or 2 OD = OG = OA.
16	Conc.	∴ the diam. of ⊙ AKGH = 2 diam. of ⊙ EDF. Q. E. D.

PROP. A.—THEOR.

A circle may be described about any regular polygon, or inscribed within it; and conversely.

E. 1	Hyp.	Let the reg. polygon be ABCDE;
2	Con. 1.	the ⊙ ABD may be circumscribed;
3	„ 2.	or ⊙ GLI may be inscribed.
C. 1	Use 9, III.	Through A, B, C desc. a ⊙, O being the centre;
2	10, I,	bis. the sides AB, BC, &c., in G, H, &c.;
3	Pst.	and join OA, OB, OC, &c.; OG, OH, OI, &c.
D. 1	C. 1, 2, & 3, III.	∵ CI = ID, IO com. and ∠ CIO = ∠ DIO;
2	4, I.	∴ OD = OC; and ⊙ passes through D;
3	*Sim.* Def. 6, IV.	∴ ⊙ also passes through E; and ∴ a ⊙ is descr. about ABCDE.
4	C. 1, 2.	Next, ∵ AG = AL, OA com. and ∠ OAG = ∠ OAL;
5	4, I. and 3, III.	∴ OG = OL, and the ⊙ touches AB and AE.
6	*Sim.*	So the ⊙ touches H, I, and K.
7	Def. 5, IV.	∴ ⊙ GLI is inscribed in the polygon.
8	Rec.	∴ *A circle may be described, &c.* Q. E. D.



USE AND APP. 1.—In a reg. polygon the angles at the centre, opposite to the sides, are all equal, and together make up four rt. angles; therefore *in constructing a polygon,* $\frac{360^\circ}{\text{No. of sides}}$ = each ∠ at the centre; and the *chord* of that ∠ = *a side;* and the *circumference* being divided *by the chord,* the points of the polygon will be obtained.

2. *To inscribe a polygon in a given circle;* divide the circle into as many equal parts as there are sides of the polygon, and join the points of the circumference by chords.

3. *To circumscribe a polygon;* draw tangents at the points where the circumference is divided into equal parts and produce them until they meet.

4. *The area of a regular polygon;* as ABCDE, is equal to the product of the perimeter into half the radius, $\frac{OL}{2}$, of the inscribed circle ;

For area of $\triangle$ OAE $=$ AE $\times$ $\frac{OL}{2}$; and there are as many triangles identical with $\triangle$ OAE as there are sides of the polygon ; $\therefore$ taking n to represent the number of sides, *area of polygon* $= n \times$ AE $\times \frac{1}{2}$ OL.

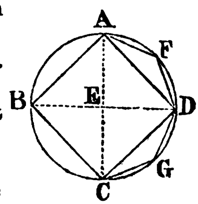
5. By an extension of the principle, as shown in 41, I. Use 4, the *area of a circle* $=$ the product of the rad. and of half the ⊙ce.

Prop 6.—Prob.

To inscribe a square in a given circle.

Con. 1, III. 11, I. Pst. 3.

Dem. Def. 15, I. Ax. 11. 4, I. 31, III. Def. 3, IV.
Def. 30, I. A square has all its sides equal and all its angles rt. angles.



E.1	Datum	Let ABCD be the given circle;
2	Quæs.	to inscribe a square in ABCD.
C.1	1, III. 11, I.	Draw the diams. AC, BD at rt. $\angle$s to each other;
2	Pst. 3.	and join AB, BC, CD, DA ;
3	Sol.	then the fig. ABCD is the □ required.
D.1	Def. 15, I.	Here EA $=$ EB $=$ EC $=$ ED;
2	D.1C.1Ax.11	$\because$ BE $=$ ED, EA com, and $\angle$s at E equal;
3	4, I. and *Sim.*	$\therefore$ BA $=$ DA; and BA $=$ DA $=$ BC $=$ CD ;
4	Conc.	$\therefore$ the quadril. ABCD is equilateral.
5	C. 1. 31, III.	Again, $\because$ BD is a diam., $\therefore$ $\angle$ BAD is a rt. $\angle$;
6	*Sim.*	also each of the $\angle$s ABC, BCD, CDA is a rt. $\angle$;
7	Conc.	$\therefore$ the qu. lat. ABCD is equiangular
8	D.4.7.Def. 30	$\therefore$ the qu. lat. is a square ;
9	Def. 3, IV.	and inscribed in the ⊙ ABCD. Q. E. F.

Cor.—The square on the radius, EC^2, of an inscribed ⊙, is half the square inscribed in the circle, and one-fourth the square on its diam., AC^2; *i. e.* $EC^2 = \frac{ABCD}{2} = \frac{AC^2}{4}$.

Sch.—In other words, the inscribed square ABCD $= 2EC^2 = \frac{1}{2} AC^2$; for the $\angle$s at E being rt. $\angle$s, we have, by 47, I, $BC^2 = BE^2 + CE^2 = 2r^2$; $\therefore BC = r\sqrt{2}$; And $AC^2 = 4EC^2 = 2BC^2$.

Ex. 1.—The radius being 8, required the area of the inscribed square.

$$2 \times 8 \times 8 = 128.$$

Ex. 2.—When the area of the insc. square is 288, what is the diam. of the circle?

$$\sqrt{(2 \times 288)} = \sqrt{576} = 24.$$

Use and App.—By bisecting the arcs AD, DC, CB, BA, and joining the points F, G, &c., an *octagon* will be formed; and by continuing the bisections, other regular polygons of 16, 32, 64, &c., sides.

Prop. 7.—Prob.

To circumscribe a square about a given circle.

Con. 1, III. 11, I. Pst. 1. 17, III.

Dem. 18, III. Def. 15-17, I. Def. 30. Def. 4, IV.

28, I. If a st. line falling upon two other st. lines makes the ext. $\angle$ = the int. and opp. $\angle$ upon the same side of the line; or makes the int. $\angle$s upon the same side together equal to two rt. $\angle$s; the two st. lines shall be parallel.

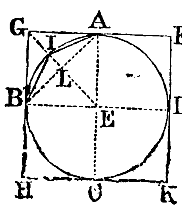
30, I. Straight lines parallel to the same straight line are parallel to each other.

Def. A. I. A parallelogram is a four-sided fig., of which the opp. sides are parallel.

34, I. The opp. sides and $\angle$s of ▱s are equal to one another, and the diam. bisects them; *i. e.* divides them into two equal parts.

Cor. 2. 46, I. Every ▱ having one rt. $\angle$ has all its $\angle$s rt. $\angle$s.

E.1	Datum.	Given a $\odot$ ABCD;
2	Quæs.	to descr. a square about it.
C.1	1, III.	Find E the cen. of the $\odot$;
2	11, I.	at E draw two diams. AC, BD, at rt. $\angle$s to each other;
3	17, III.	and through A, B, C, D, tangents forming fig. GHKF;
4	Sol.	then GHKF is the □ required.



D.1	C. 3 and 2.	$\because$ GF is a tang. at A, and AE a st. line from A to cen. E;
2	18, III.	$\therefore$ the $\angle$s at A are rt. $\angle$s;
3	*Sim.*	So the $\angle$s at B, C, D, are rt. $\angle$s.
4	C. 2 & 3. 28, I.	And $\because$ the $\angle$s AEB, EBG, are rt. $\angle$s, $\therefore$ GH $\parallel$ AC;
5	*Sim.* 30, I.	also FK $\parallel$ AC, and GH $\parallel$ FK;
6	*Sim.*	and GF, FK, each $\parallel$ BD;
7	Def. A.I.	$\therefore$ the figures GK, GC, AK, FB, BK, are ▱s;
8	34, I.	and $\therefore$ GF = HK and GH = FK;
9	Def.15-17. 34, I.	and $\because$ AC = BD; AC = GH = FK; and BD = GF = HK;
10	Conc.	$\therefore$ GH = FK = GF = HK, and FGHK is equilateral.
11	D. 7. C. 2.	Again, $\because$ GBEA is a ▱, and $\angle$ AEB a rt. $\angle$;
12	34, I.	$\therefore$ $\angle$ AGB is a rt. $\angle$;
13	Cor. 2. 46, I.	and $\therefore$ $\angle$s at H, K, and F are rt $\angle$s;
14	Conc.	$\therefore$ FGHK is rectangular;
15	Def.10,14.Def.30	$\therefore$ the fig. FGHK is a square,
16	Def. 4, IV.	and it is circumscribed about the $\odot$ ABC.

Q. E. F.

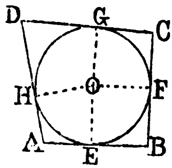
Cor.—In the same circle the circumscribed square GHFK is double of the inscribed square; *i. e.* BD^2 or $AC^2 = 2AB^2 = 4\,AE^2$.

Use and App. 1.—By bisecting the arcs and drawing chords a *regular octagon* may be *inscribed*, AI, IB, &c.; and by drawing tangents through the angular points of the inscribed octagon, a *reg. octagon* may be *circumscribed.*

2.—A reg. octagon, AIB, &c., inscribed in a circle, ABCD, is equal to the rectangle under the sides, AB, BD, of the inscribed and circumscribing squares.

D.1	6, 7, IV.	For BD and AE are at rt. $\angle$s; and BD $\parallel$ GF $\parallel$ HK side of circumscribing $\square$;
2	28, III.	also AI = IB, and arc AI = arc IB; and EI $\perp$ AB bisects it in I;
3	41, I.	$\therefore 2\triangle$ EAI = IE $\cdot$ AL; and 2 $\triangle$ EBI = IE $\cdot$ BL.
4	*Add.*	$\therefore$ 2 the qu. lat. BEAI = IE $\cdot$ (AL + BL) = IE $\cdot$ AB;
5		Or 4 BEAI = 2 BE $\cdot$ BA = BD $\cdot$ BA;
6	Remk.	Now the octagon is 4BEAI; and BD = side of circumscrg. $\square$, and AB = side of inscribed square.

3.—If a quadril. ABCD be circumscribed about a circle EFGH, any two of its opp. sides AB+CD, or AD+BC = half its perimeter, *i. e.* $\frac{AB+BC+CD+AD.}{2}$



D.1	Cor. 37, III.	∵ tangs. AE = AH, BE = BF, CF = CG, and DG = DH;
2	Ax. 1.	∴ AE + BE + CF + DG = AH + BF + CG + DH;
3		*i. e.* AB + CD = AD + BC, half the perimeter.

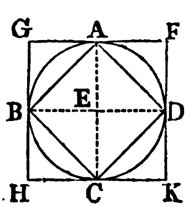
Prop. 8.—Prob.

To inscribe a circle in a given square.

Con. 10, I. 31, I. Through a given point to draw a parallel to a given st. line.

Dem. Def. 30, I. Cor. 2. 46, I. 34, I. Def. 5, IV.
Ax. 7. Things which are halves of the same are equal to one another.
29, I. If a line fall upon two par. st. lines, it makes the alt. $\angle$s equal to one another; and the ext. $\angle$ = int. and opp. $\angle$ upon the same side; and likewise the two int. $\angle$s upon the same side together = two rt. $\angle$s.

Cor. 1. 16, III. If a st. line be drawn at rt. $\angle$s to any diam. of a $\odot$ from its extremity, it shall touch the $\odot$ at the extremity; and a st. line touching the $\odot$ at one point shall touch it at no other point.



E.1	Datum.	Let GHKF be the given square,
2	Quæs.	to inscr. a circle in that $\square$.
C.1	10, I.	Bis. the sides GH, GF in B and A;
2	31, I.	through A and B draw AC $\parallel$ GH or FK, and BD $\parallel$ GF or HK;
3	Pst. 3.	with rad. EA, from E, draw a $\odot$;
4	Sol.	the $\odot$ ABCD is inscr. in the sq. GHKF.

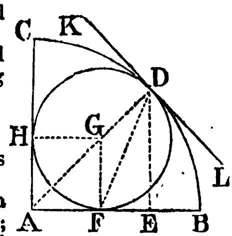
D.1	C. 2.	Each of the figs. GD, DH, GC, CF, GE, EK, HE, EF is a □;
2	Def. 30.	and ∵ each contains an ∠ of the sq. GHKF;
3	Cor. 2. 46, I.	∴ each of those figures is rectangular;
4	34, I.	and ∴ of each, the opp. sides are equal.
5	Def. 30. E 1.	Now ∵ GF = GH, and GA = ½GF, and GB = ½GH;
6	Ax. 7. D. 3.	∴ GA = GB, and BE = EA;
7	*Sim.*	Thus EA = ED = EC = EB;
8	Conc.	∴ the ⊙ from E, with rad. EA, passes through the ·s B, C, D.
9	29, I.	And ∵ the ∠s at A, B, C, D, are rt. ∠s;
10	C.	and the st. lines GF, FK, KH, HG are at the ends of diams. BD, AC;
11	Cor. 16, III.	∴ each of those st. lines is a tang to the ⊙;
12	Def. 5, IV.	∴ the ⊙ ABCD is inscribed in the □ GHKF.

Q. E. F.

N.B.—The diagram will illustrate the Cor. to Pr. 7, IV.

SCH.—Euclid confines himself in this book to the inscription and circumscription of circles and regular rt. lined figures,—but circles may be inscribed in segments and sectors; for example,

To inscribe a circle in a given quadrant ABC.



C.1	9, I. 31, I.	With AD bis. ∠ CAB and draw DE ∥ AC;
2	9, I. 31, I.	with DF bis. ∠ ADE, and draw FG ∥ AC and meeting AD in G;
3	31, I. 11, I.	also draw GH ∥ AB and KL ⊥ AD in D;
4	Sol.	the cen. of the required ⊙ is G; and GD or GF its rad.
D.1	C. 1. H.	∵ ∠ DAB = ½ ∠ CAB a rt. ∠, and GFA is a rt. ∠;
2	6, I.	∴ ∠ AGF = ½ a rt. ∠, and AF = FG;
3	34, I. Ax. 1.	but AF = HG; ∴ FG = HG.
4	C. 2. 29, I.	Again ∵ ∠ GDF = ∠ EDF, and ∠ EDF = ∠ DFG;
5	Ax. 1. 6, I. D. 3.	∴ ∠ GDF = ∠ DFG, and GF = GD = HG;
6	9, III.	∴ the ⊙ HFD passes through the points, D, F, H.
7	Cor. 46, I.	Also ∵ the ∠s at H and F are rt. ∠s;
8	16, III.	∴ the ⊙ touches AC and AB in H and F;
9	11, III.	and ∴ AG, joining the centres A and G, passes through D;
10	C. 3 and D. 6.	and ∵ LD or KD, a perp. to AD at D, is a tang. to arc CDB and to the ⊙ DFH;
11	11, III.	∴ the ⊙ DFH touches the arc CDB.

Q. E. F.

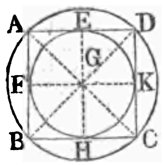
PROP. 9.—PROB.

To circumscribe a circle about a given square.

CON. Pst. 1 and 3.

DEM. Def. 30, I. Axs. 11 & 7. Def. 6, IV. 8, I. If the △s have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the ∠ which is contained by the two sides of the one, shall be equal to the ∠ contained by the two sides equal to them of the other.

6, I. If two ∠s of a △ be equal to one another, the sides opp. to the equal ∠s shall be equal to one another.



E.1	Datum.	Let ABCD be the given □;
2	Quæs.	to circumsc. a ⊙ about ABCD.
C.1	Pst. 1. 3.	Join AC, BD cutting in G; and with GA desc. a ⊙;
2	Sol.	*that* ⊙ ABCD is the ⊙ required.
D.1	Def. 30, I.	∵ in △s ABC, ADC, DA = AB, CB = CD, and AC com.;
2	8, 1.	∴ ∠ DAC = ∠ BAC, *i. e.* ∠ BAD is bisd. by AC;
3	*Sim.*	So ∠s ABC, BCD and CDA are bisected by BD and AC.
4	Def. 30, I.	Now ∵ ∠ DAB = ∠ ABC;
5	D. 3.	and ∠ GAB = ½∠ DAB, and ∠ GBA = ½∠ ABC;
6	Ax. 7 & 6, I.	∴ ∠ GAB = ∠ GBA, and GA = GB.
7	*Sim.*	So GA = GB, and GC = GB = GD;
8	Ax. 1.	∴ GA = GB = GC = GD.
9	9, III.	Hence a ⊙ from G, with rad. GA, will pass through B, C, and D.
10	D. 8.	And ∵ the ⊙ ABC passes through the angular • s of the sq. ABCD,
11	Def. 6, IV.	∴ the ⊙ ABC is c. scrd. about the □ABCD.

Q. E. F.

N.B. In the diagram a circle is also inscribed in the square ABCD.

Prop. 10.—Prob.

To construct an isosceles triangle, having each of the angles at the base double of the third, or vertical angle.

Con. Pst. 3. 1. IV. Pst. 1. 5, IV. 11, II. To divide a given line into two parts, so that the rect. contained by the whole and one of the parts shall equal the sq. of the other part.

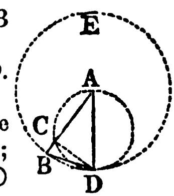
Dem. Ax. 1. 32, III. 6, I. 37, III. If from a · *without* a ⊙ there be drawn two lines, one of a which cuts the ⊙, and the other meets it; if the rect. contained by the whole line which cuts the ⊙, and the part of it without the circle, be equal to the sq. of the line which meets it, the line which meets shall touch the ⊙.

Ax. 2. If equals be added to equals the wholes are equal.

32, I. If a side of a △ be produced, the ext. ∠ = the two int. and opp. ∠s, and the three int. ∠s of every △ are together equal to two rt. ∠s.

Def. 25, I. An Isosceles △ is that which has two equal sides or legs.

5, I. The ∠s at the base of an isosc. △ are equal to each other; and if the eq. sides be produced, the ∠s on the other side of the base shall be equal.



E.1	Data.	Given the ∠s ABD, ADB each = 2 ∠ BAD;
2	Quæs.	to construct an isosc. △ ABD.
C.1	11, II.	Take any line AB and divide it so that AB · BC = AC²;
2	Pst. 3.	from A with rad. AB desc. ⊙ BDE;
3	1, IV. Pst. 1.	in ⊙ BDE place BD = AC ≯ diam. of ⊙ BDE, and join AD;
4	Sol.	then in △ ABD, ∠ ABD = ∠ BDA = 2 ∠ BAD.
5	Pst. 1. 5., IV.	Join DC, and about △ ADC desc. ⊙ ACD.
D.1	C. 1. Ax. 1.	∵ AB · BC = AC², and AC = BD; ∴ AB · BC = BD²;
2	C.	and ∵ from B, a · out of ⊙ ACD, are drawn BCA, BD, one cutting the ⊙ in C, the other meeting it in D;
3	D. 1. 37, III.	and ∵ AB · BC = BD², ∴ BD touches the ⊙ ACD in D.

D.4	D. 3. & C.	Again ∵ BD touches the ⊙ ACD, and DC a st. line from the • of contact D, cuts that ⊙;
5	32, III.	∴ ∠ BDC = ∠ DAC in the alt. seg.;
6	*Add.* Ax. 3.	Add ∠ CDA to each; ∴ ∠ BDA = ∠ CDA + ∠ DAC;
7	32, I. Ax. 1.	But ∵ ext. ∠ BCD = ∠s CDA, DAC, ∴ ∠ BDA = ∠ BCD;
8	C. 5, I. Ax. 1.	and ∵ AD = AB, and ∠ BDA = ∠ CBD; ∴ ∠ CBD or DBA = ∠ BCD;
9	Ax. 1.	∴ ∠ BDA = ∠ DBA = ∠ BCD.
10	D. 7. 9. 6, I.	Again ∵ ∠ DBC = ∠ BCD; ∴ BD = DC;
11	C.3. Ax.1.5,I.	but BD = CA, ∴ AC = CD and ∠ CDA = ∠ DAC;
12	Ax. 2.	∴ ∠s CDA + DAC = 2 ∠ DAC;
13	D. 7. 32, I.	but ∠ BCD = ∠ CDA + ∠ DAC; ∴ ∠ BCD = 2 ∠ DAC.
14	D. 9.	Now ∠ BCD = ∠ BDA = ∠ DBA;
15		∴ ∠ BDA = DBA = 2 ∠ DAB.
16	Rec.	∴ the △ ABD is the isosc. △ required.

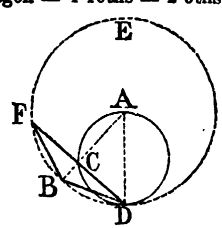
Q. E. F.

USE AND APP.—The following are some of the various problems which bear a close relation to the 10th :—

1°. *The side* AC *inscribed in the smaller* ⊙ ACD *equals the side of a regular* pentagon *in that circle, and also equals the side of a regular* decagon *in the larger* ⊙ BDE.

Prel. 1	2 Cor. 15, I.	∵ the ∠s formed by lines from a central point = 4 rt. ∠s;
2		and ∵ in a regular polygon the central ∠s are all equal;
3		∴ the central ∠ of a pent. = 4-5ths of a rt. ∠;
4		and ∴ the central ∠ of a decagon = 4-10ths = 2-5ths of a rt. ∠.

First—For the *decagon* in ⊙ DEF.,



D.1	10, IV. Prel. 4.	∵ AC = BD; and ∠ A = 2-5ths of a rt. ∠;
2	Conc.	∴ AC or BD = the side of a reg. decagon in ⊙ BEF.

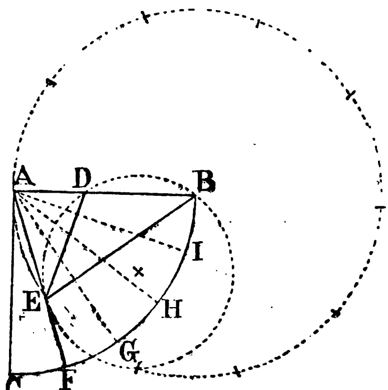
Second,—For the *pentagon* in ⊙ ACD.

D.1	10, IV., & C.	∵ CD = BD = AC; and AC subtends ∠ ADC at the ⊙ce of a ⊙ ACD,
2	20, III.	and ∵ also ∠ ADC = 2-5ths of a rt. ∠ = ½ the ∠ at the centre of ⊙ ACD;
3	Ax. 6.	∴ the ∠ at the cen. of ⊙ ACD = 4-5ths of a rt. ∠;
4	Conc.	and ∴ AC = side of a reg. pent. in the ⊙ ACD.

2°. *On the side* DC *being produced to meet the circle* BDE *in* F, *and* FB *being joined, the* ∠ ABF = *three times* ∠ BFD.

D.1	C. and 32, I.	∵ ∠ BAD = ∠ ADC, and ∵ ∠ BCD = ∠s BAD + ADC;
2	Ax. 1.	∴ ∠ BCD = 2 ∠ BAD.
3	20, III.	Now, ∠ BAD = 2 ∠ BFD; and ext. ∠ BCD = 4 ∠ BFD = ∠s ABF + BFD;
4	*Sum.*	from each side take away ∠ BFD;
5	Ax. 3.	∴ ∠ ABF = 3 ∠ BFD.

3°. *To quinquisect*, i.e., *divide a quadrant* BAC *into five equal parts.*



C.1	10, IV.	Construct an isosc. △ BAE, having ∠ BAE = ∠ BEA = 2 ∠ ABE;
2	9, I. Sch.	prod. AE to F, and div. ∠ FAB into 4 equal parts;
3	Sol.	then ∠ CAF = ∠ FAG = ∠ GAH = ∠ HAI = ∠ IAB = 1-5th ∠ BAC.
D.1	C. & Def.	∵ of a decagon the central ∠ ABE = 2-5ths of a rt. ∠;
2	C.	and ∵ ∠ BAF = 2 ∠ ABE = 4-5ths of a rt. ∠;
3	Datum.	but ∠ BAC is a rt. ∠;
4	Ax.	∴ ∠ CAF = 1-5th of a rt. ∠, and ∠ FAB = 4-5ths of a rt. ∠.
5	C. 2.	and ∠ FAB is div. into four equal parts;
6	Conc.	∴ the quadrant BAC has been divided into five equal parts.

Q. E. F.

It is evident that, by the process of continual bisection, the quadrant may now be divided into 10, 20, 40, 80, &c., equal parts.

Prop. 11.—Prob.

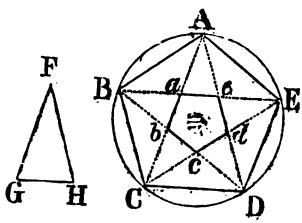
To inscribe an equilateral and equiangular pentagon in a given circle.

Con. 10, IV. 2, IV. 9, I. Pst. 1.

Dem. 26, III. In eq. circles, eq. angles stand upon eq. arcs, whether they be at the centres or circumferences.

29, III. In eq. circles eq. arcs are subtended by eq. st. lines. Ax. 2.

27, III. In eq. circles, the angles which stand upon eq. arcs are equal to one another, whether they be at the centres or the ⊙ces. Def. 3, IV.

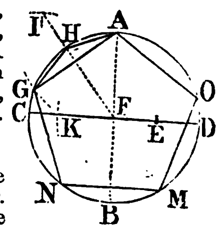


E.1	Dat.	Let ABCDE be the given ⊙;
2	Quæs.	to inscr. therein a reg. pent.
C.1	10, IV.	Desc. an isosc. △ FGH, with ∠ G = ∠ H = 2 ∠ F;
2	2, IV.	in ⊙ ABCDE insc. △ ACD equ. ang. to △ FGH; so that ∠ CAD = ∠ F, and ∠s ACD, CDA each = ∠ G = ∠ H;
3	9, I,	Bis. ∠s ACD, ADC by CE, CB cutting the ⊙ce in B and E;
4	Pst. 1.	join AB, BC, DE and EA;
5	Sol.	then the fig. ABCDE is the pent. required.
D.1	C. 2, & 1.	∵ △ ACD is eq. ang. to △ FGH, and ∠ G = ∠ H = 2 ∠ F;
2	Ax. 1,	∴ ∠ ACD = ∠ ADC = 2 ∠ CAD.
3	C. 3.	and ∵ the eq. ∠ s ACD, ADC are bisected by CE, DB;
4	Ax. 7.	∴ ∠ DAC = ∠ ACE = ∠ ECD = ∠ CDB = ∠ BDA;
5	26, III.	but eq. ∠s at the ⊙ce stand on eq. arcs;
6	Conc.	∴ the arcs, AB, BC, CD, DE, EA are all equal;
7	29, III.	and eq. arcs have equal chords;
8	Conc.	∴ the chords AB, BC, CD, DE, EA are all equal;
9	Conc. 1.	∴ the pentagon is equilateral.
10	*Add.*	Again, to each of the eq. arcs AE, DE, add arc BCD;

11	Ax. 2.	$\therefore$ the whole arc ABCD = the whole arc EDCB;
12	C.&D.10&11	but $\angle$ AED stands on arc ABCD, and $\angle$ BAE on arc EDCB;
13	27, III	$\therefore$ $\angle$ BAE = $\angle$ AED;
14	*Sim.*	So $\angle$s ABC, BCD, CDE, each = $\angle$ BAE, or $\angle$ AED;
15	Conc. 2.	$\therefore$ the pent. is equiangular.
16	D. 9.	And it has been shown to be equilateral;
17	C.	and $\because$ the angular $\cdot$s A, B, C, D, E are in the $\odot$ce,
18	Def. 3, IV.	$\therefore$ the pent. is inscribed in the circle.

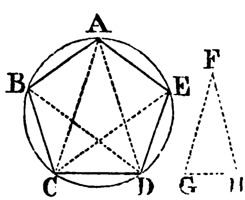
Q. E. F.

SCH. I.—In a reg. pentagon we may remark; 1st., that each diagonal, as AC, is parallel to the side, as ED, to which it is not conterminous; 2nd., that triangles, BC*b*, ED*e*, AE*d*, &c., are isosc. $\triangle$s, equiangular with $\triangle$ CAP, &c., and having the $\angle$ at the base = *twice* the vert. angle; 3rd., that the fig. AB*c*E is a lozenge, also BCD*e*, &c.; and 4th., that fig. *abcde* is a reg. pentagon. LARDNER'S *Euclid*, p. 130.



II. *Practically*, a pentagon is inscribed in a circle, by drawing two perpendicular diameters, AB, CD, and bisecting the rad. FD in E; from E with EA desc. AK, and from A with AK desc. KG; then AG is the side of the pentagon; and if the arc AG, be bisected in H, the chord AH is a side of a decagon. See EUCLID 10, xiii.

III. It is generally *by aid of isosc.* $\triangle$s, with the $\angle$s at the base *equimultiples* of the vert. $\angle$, that reg. rectil. figures which have an *odd* number of sides are inscribed in circles. Thus, *for the pentagon;*



By 10, iv., desc. an isosc. $\triangle$ FGH, having each of the $\angle$s, G and H at the base double of $\angle$ F, the $\angle$ at the vertex.

2, iv., in the given $\odot$ insc. a $\triangle$ ACD eq. ang. with $\triangle$ FGH;

9, I., bis. the $\angle$s ACD, ADC; and let the bisecting lines be produced to meet the $\odot$ce in B and E;

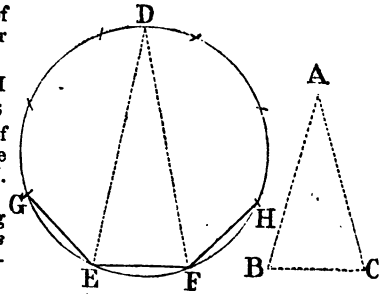
Then the points, A, B, C, D, E, are the angular points of the required pentagon.

IV. Reg. rectil. polygons with an *even* number of sides may also be inscribed in circles *by aid of isosc* $\triangle$s; but in such isosc. $\triangle$s, the $\angle$s at the base are multiples *sesquialter*, as it is termed, of the vert. $\angle$; *i. e.*, each angle at the base contains $1\frac{1}{2}$ times, or $2\frac{1}{2}$ times, or $3\frac{1}{2}$ times, &c., the magnitude of he vert. $\angle$. Thus *for the octagon*;

Construct an isosc. △ ABC, of which each ∠ at the base, ∠ B or ∠ C = $2\frac{1}{2}$ times the vert. ∠ A.

By 2, iv., in the given ⊙ DGH insc. a △DEF eq. ang. with △ ABC;

then the base EF, the chord of the arc EF, will be a side of the octagon to be inscribed in ⊙ DGH.



V. *Formulæ* for determining *the relative magnitudes of the angles of isosc.* △*s,* to be used *in the construction of regular polygons.*

The vert. ∠ of such isosc △s = $\dfrac{180^\circ}{\textit{number of sides}}$

The multiplier of that vert. ∠ for the ∠s at the base $= \dfrac{\text{sides} - 1}{2.}$

Thus in a regular polygon of

5	sides, the vert. ∠	$=36^\circ$;	the multiplier is	2;	the ∠s at the base each	$=72^\circ$
7	,,	$=25^\circ\frac{5}{7}$;	,,	3;	,,	$=77^\circ\frac{1}{7}$
9	,,	$=20^\circ$;	,,	4;	,,	$=80^\circ$
11	,,	$=16^\circ\frac{4}{11}$;	,,	5;	,,	$=81^\circ\frac{9}{11}$
13	,,	$=13^\circ\frac{11}{13}$;	,,	6;	,,	$=83^\circ\frac{1}{13}$

And in a regular polygon of

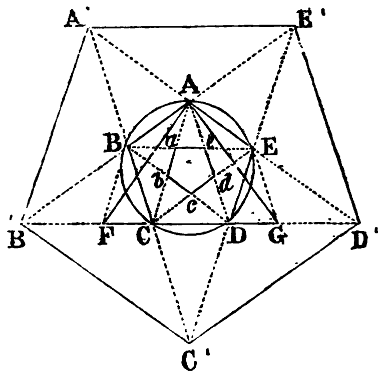
6	sides, the vert. ∠	$=30^\circ$;	the multiplier is	$2\frac{1}{2}$;	the ∠s at the base each	$=75^\circ$
8	,,	$=22^\circ\frac{1}{2}$;	,,	$3\frac{1}{2}$;	,,	$=78^\circ\frac{3}{4}$
10	,,	$=18^\circ$;	,,	$4\frac{1}{2}$;	,,	$=81^\circ$
12	,,	$=15^\circ$;	,,	$5\frac{1}{2}$;	,,	$=82^\circ\frac{1}{2}$
14	,,	$=12^\circ\frac{6}{7}$;	,,	$6\frac{1}{2}$;	,,	$=83^\circ\frac{4}{7}$
16	,,	$=11^\circ\frac{1}{4}$;	,,	$7\frac{1}{2}$;	,,	$=84^\circ\frac{3}{8}$
18	,,	$=10^\circ$;	,,	$8\frac{1}{2}$;	,,	$=85^\circ$
20	,,	$=9^\circ$;	,,	$9\frac{1}{2}$;	,,	$=85^\circ\frac{1}{2}$

APP.—I. *To draw a triangle equal in area to a given pentagon,* ABCDE see on page 118.

C.1	Pst. 1 & 2	Join AC, AD, and prod. CD;
2	31, I.	through B and E draw BF ∥ AC, EG ∥ AD;
3	Pst. 1.	and join AF, AG;
4	Sol.	then △ AFG = pent. ABCDE.
D.1	37, I.	∵ △ ACF = △ ACB, and △ ADG = △ ADE;
2	*Add.*	add △ ACD to the equals;
3	Ax. 2.	∴ △ AFG = pent. ABCDE.

II. *The lines* AC, BD, CE, DA, EB, *joining the alternate angles,* A, C; B, D; C, E; D, A; E, B; *of a reg. pentagon,* ABCDE, *will form another reg. pentagon, a b c d e; and the points of intersection,* A′, B′, C′, D′, E′, *of the alternate sides of* ABCDE *produced, namely,* AB *and* CD, *or* DE; BC *and* AE, *or* ED; CD *and* AB, *or* AE; DE *and* AB, *or* BC; EA *and* BC, *or* CD; *will also form another reg. pentagon,* A′ B′ C′ D′ E′.

First—The figure $a\,b\,c\,d\,e$ is a regular pentagon.



D.1	C. & H.	∵ arc BCDE = arc AEDC, and arc BC = arc AE;
2	27, III.	∴ ∠ BAE = ∠ ABC, and ∠ BAC = ∠ ABE;
3	Ax. 3.	∴ remg. ∠ CAE = remg. ∠ EBC.
4	H.15, I. & D.3.	and ∵ BC = AE, ∠ B a C = ∠ A a E, and ∠ CAE = ∠ EBC;
5	26, I. 6, I.	∴ A a = B a, and ∠ ABE = ∠ BAC;
6	32, I.	∴ ∠ A a e = 2 ∠ ABE = 2 ∠ AEB = ∠ A e a;
7		and ∴ A a = A e = B a = E e.
8	Remk.	Thus the △s A ae, B a b, C b c, &c., are isosc. △s;
9		they are also equal; ∴ $ae = ab = bc = cd = dc$.
10	5, I.	Now, in isosc. △s the ∠s on the other side of the base are equal;
11		∴ ∠ eab = ∠ abc = ∠ bcd, &c.
12		∴ also the fig. $a\,b\,c\,d\,e$ is equiangular.
13	D. 9 & 12.	∴ $a\,b\,c\,d\,e$ is a regular pentagon. Q. E. F.

Second,—Also A′ B′ C′ D′ E′ is a reg. pentagon.

D.1	H. 13, I.	∵ ∠ ABC = ∠ BAE, and ∠s A′BA + ABC = ∠s A′AB + BAE;
2	Ax. 3. *Sim.*	∴ ∠ A′AB = ∠ A′BA; and so = ∠ B′CB = ∠ B′BC, &c.:
3	H. 26, I.	Also AB = BC; ∴ △ ABA′ = △ BCB′;
4	*Sim.*	So △s CDC′, DED′ and EAE′ are equal.
5	D. 1-4.	And ∵ BB′, BA′ = E′A, AA′, & ∠ B′BA′ = ∠ E′AA′;
6	4, I. & *Sim.*	∴ B′A′ = A′E′; and so = E′D′ = D′C′ = C′B′;
7	Conc.	∴ the fig. A′B′C′D′E′ is equilateral.
8	D.	Also the 3 ∠s at A′ = the 3 ∠s at B′ = 3 ∠s at C′, &c.
9	Conc.	∴ the fig. A′B′C′D′E′ is equiangular;
10	D. 7, & 9.	∴ it is a reg. pentagon. Q. E. F.

Prop. 12.—Prob.

To circumscribe an equilateral and equiangular pentagon about a given circle.

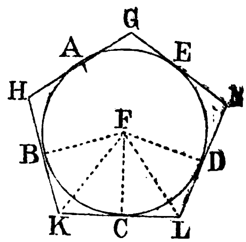
Con. 11, IV. 17, III. 1, III. Pst. 1

Dem. Ax. 1. Def. 15, I. Ax. 3. 8, I. 4, I. 27, III. Ax. 7. Def. 10, I. 26, I. Ax. 6. Def. 4, IV.

28, III. In equal circles, equal st. lines cut off eq. arcs, the gr. = the gr., and the less to the less.

18, III. If a st. line touches a ⊙, the st. line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

47, I. In any rt. angled △ the square on the side opp. to the rt. ∠ is equal to the squares on the sides containing the rt. ∠.



E.1	Dat.	Let A B D be the given ⊙;
2	Ques.	to desc. a reg. pent. about that ⊙.
C.1	11, IV.	In ⊙ ABD insc. a reg. pent., of which the angular • s are A, B, C, D, E, and the arcs, AB, BC, CD, DE, EA, are equal;
2	17, III.	through A, B, C, &c., draw GH, HK, KL, &c., tangents to the ⊙ ABD;
3	Sol.	then the fig. GHKLM is the pent required.
4	1, III.	Find F the centre of the ⊙ ABD;
5	Pst. 1.	and join FB, FK, FC, FL, and FD.
D.1	C. 1.	∵ A, B, C, D, E, are angular • s of the reg. pent.
2	28, III.	∴ arcs AB, BC, CD, &c., are equal;
3	C. 2, 4.	and ∵ KL is a tang.; and from cen. F, FC is drawn to C;
4	18, III.	∴ FC ⊥ KL, and ∠s FCK, FCL, rt. ∠s.
5	*Sim*	So ∠s FBK, FDL, &c., are rt ∠s;
6	D. 4 & 5.	Now ∵ ∠s FCK and FBK are rt. ∠s;
7	47, I.	∴ $FC^2 + CK^2 = FK^2$, and $FB^2 + BK^2 = FK^2$;
8	Ax. 1.	∴ $FC^2 + CK^2 = FB^2 + BK^2$;
9	Def. 15. Ax. 3.	but $FC^2 = FB^2$; ∴ $CK^2 = BK^2$, and CK = BK.
10	C. and D. 9.	Also ∵ FC = FB, FK com., and CK = BK;
11	8, I.	∴ ∠ CFK = ∠ BFK, and ∠ CKF = ∠ BKF;
12	Remk.	thus FK bisects ∠s BFC and BKC, and ∠ BFC = 2 ∠ BFK, and ∠ BKC = 2 ∠ BKF.

13	*Sim.*	So FL bisects ∠s CFD, CLD.
14	C.1.27,III.Ax.7	Again ∵ arc B C = arc C D, ∴ ∠ BFC = ∠ CFD, and ∠ KFC = ∠ CFL.
15	Def. 10. D. 14.	Also ∵ ∠ KCF = ∠ LCF, FC com., and ∠ KFC = ∠ CFL;
16	26, I.	∵ KC = CL, and ∠ FKC = ∠ FLC.
17	*Sim.*	Now ∵ KC = CL; ∴ KL = 2 KC, and so HK = 2 BK;
18	D.9. D.17. Ax.6.	and ∵ BK = KC, KL = 2 KC, and HK = 2 BK; ∴ HK = KL.
19	*Sim.*	So GH, GM, ML, each = HK, or KL;
20	Conc.	∴ the pent. GHKLM is equilateral.
21	D. 16. 12. 13.	*Lastly*, ∴ ∠ FKC = ∠ FLC, ∠ HKL = 2 ∠ FKC, and ∠ KLM = 2 ∠ FLC;
22	Ax. 6.	∴ ∠ HKL = ∠ KLM;
23	*Sim.*	and so ∠s KHG, HGM, GML, each = ∠ HKL or ∠ KLM;
24	Conc.	∴ the pent. GHKLM is equiangular;
25	D. 20 & 24.	∴ the fig. GHKLM is a reg. pentagon.
26	C. 2.	And each side touches the given ⊙ ABD;
27	Def. 4, IV.	∴ the pent. GHKLM is circumscribed about the ⊙ ABD. Q. E. F.

SCH.—It is a general truth, that, "If the circumference of a circle be divided into any number of parts, the chords joining the points o division shall include a regular polygon, inscribed in the circle; and the tangents drawn through those points shall include a regular polygon of the same number of sides circumscribed about the circle."

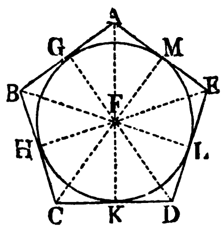
PROP. 13.—PROB.

To inscribe a circle in a given equilateral and equiangular pentagon.

CON.—9, I. 12, I. Pst. 3.

DEM.—4, I. Ax. 1. Ax. 11. All rt. ∠s are equal to one another. 26, I. Cor. 16. III. Def. 5, IV.

E.	1	Datum.	Let ABCDE be the given reg. pentagon;
	2	Quæs.	in that pent. to inscribe a circle.



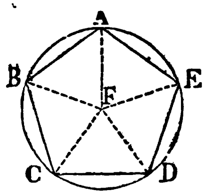
C. 1	9, I.	Bis. $\angle$s BCD, CDE by CF, DF cutting in F,
2	12, I.	from F draw FG, FH, FK, &c., perps. to AB, BC, &c.;
3	Pst. 1.	join FA, FB, FC, FD, FE;
4	Pst. 3.	with any of the perps. from F, desc. a $\odot$ GHL;
5	Sol.	then the $\odot$ GHL is the circle required.
D. 1	H. C. 1.	$\because$ in $\triangle$s BCF, DCF, BC = CD, CF, com., and $\angle$ BCF = $\angle$ DCF;
2	4, I.	$\therefore$ BF = FD, and $\angle$ CBF = $\angle$ CDF;
3	C. 1.	But $\angle$ CDF = $\frac{1}{2}$ $\angle$ CDE, and $\angle$ CDE = $\angle$ CBA;
4	Ax. 1.	$\therefore$ $\angle$ CDF = $\frac{1}{2}$ $\angle$ CBA;
5	D. 2 Ax. 1.	and $\angle$ CBF = $\angle$ CDF; $\therefore$ $\angle$ CBF = $\frac{1}{2}$ $\angle$ CBA, *i. e.*, BF bisects $\angle$ CBA;
6	*Sim.*	So $\angle$s BAE, AED are bisected by AF, FE.
7	C. 1, 2.	Now in $\triangle$s FHC, FKC, $\because$ $\angle$ HCF = $\angle$ KCF, $\angle$ FHC = $\angle$ FKC and FC com.;
8	26, I.	$\therefore$ perp. FH = perp. FK;
9	*Sim.*	also FK = FL = FM = FG;
10	D. 8 & 9.	$\therefore$ the five st. lines are equal;
11	Remk.	and a $\odot$ from F at the distance of one of the five lines, FG, will pass through H, K, L, M;
12	Cor. 16, III.	that $\odot$ GHKLM will also touch the st. lines;
13	Def. 5, IV.	$\therefore$ $\odot$ GHL is inscribed in the given pent. ABCDE. Q. E. F.

Prop. 14.—Prob.

To circumscribe a circle about a given equilateral and equiangular pentagon.

Con. 9, I. Psts. 1 and 3.

Dem. 13, V. Ax. 7. 6, I. Def. 6, IV.



E.1	Datum.	Let A B C D E be a reg. pent.;
2	Quæs.	to circumsc. a $\odot$ about that pent.
C.1	9, I.	Bis. the $\angle$s B C D, C D E, by C F, F D;
2	Pst. 1.	from F, the point in which C F, F D, meet, draw F B, F A, F E, to the points B, A, E;
3	Pst. 3.	with any one, as F A, desc. a $\odot$ A C E;
4	Sol.	then the $\odot$ A C E is circumsc. about the pent. A B C D E.
D.1	13, IV. *Sim.*	As in 13, IV., $\angle$s C B A, B A E, A E D, are bis. by F B, F A, F E;
2	H. & C. 1.	and $\because$ $\angle$ B C D $=$ $\angle$ C D E, and $\angle$ F C D $= \frac{1}{2}$ $\angle$ B C D, and $\angle$ C D F $= \frac{1}{2}$ $\angle$ C D E;
3	Ax. 7 & 6. I.	$\therefore$ $\angle$ F C D $=$ $\angle$ F D C, and C F $=$ D F.
4	*Sim.*	So F B, F A, F E, each $=$ F C $=$ F D;
5	Ax. 1.	$\therefore$ F A $=$ F B $=$ F C $=$ F D $=$ F E.
6	Remk.	Hence, a $\odot$ from F, with any one of these sides, as F A for a rad., will pass through the extremities A, B, C, D, E;
7	D. 6.	and $\therefore$ the $\odot$ A C E passes through the angr. $\cdot$ s A, B, C, D, E;
8	Def. 6, IV.	$\therefore$ the $\odot$ A C E is circumscd. about the reg. pent. A B C D E. Q. E. F.

SCH.—I. Generally the method of bisecting two adjoining angles of any reg. polygon may be applied to the circumscribing and inscribing of a circle about, or in that polygon; for, "If any two adjoining angles, as B C D, C D E, of a regular polygon, A B C D E, be bisected, the intersection F of the bisecting lines, C F, D F, shall be the common centre of two circles, the one circumscribing the polygon, the other inscribed in it."

II. Propositions 12, 13, and 14, may be generalised so as to include all regular polygons, and will then embrace the three problems :—

PROB. 1.—*To circumscribe a regular polygon about a given circle.*

To be done by drawing tangents through the angular points of the inscribed polygon.

PROB. 2.—*To inscribe a circle in a regular polygon.*

By bisecting two adjacent angles, and from the point where the angle-bisecting lines meet, drawing a perpendicular to a side of the polygon, and, with that perpendicular for radius, and this point as centre, by describing a circle.

PROB. 3.—*To circumscribe a circle about a given polygon.*

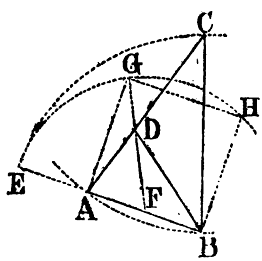
By bisecting two adjacent angles, and from the point where the angle-bisecting lines meet, with one of those lines for radius, describing a circle.—See HOSE's Euclid, p. 153.

ADDENDA TO 14, IV.

Before dismissing the *regular pentagon* and its intimately allied figure, the *regular decagon*, it will be useful to state some of the *principles* of their construction.

I. To analyze the conditions on which the drawing of those figures depends,

First—Of the *Decagon* :



At the centre there are *ten* equal angles, each = 1-10th of 360°, or 4-10ths = 2-5ths of 90°. At C draw an ∠ equal 2-5ths of a rt. ∠; and from C, with any rad., make CA = CB, and join A B; then AB the chord of 1-10th of 360° will measure the circle into *ten* equal parts, and consequently is the side of a regular decagon.

Since ∠ C = 1-5th of 180°, and ∠ A = ∠ B, ∴ ∠ A and ∠ B of △ A B C each = 2-5ths of 180°, and each is double of ∠ C.

Bis. ∠ A B C, and the △ B D C is an isosc. △ and the ext. ∠ A D B = 2 ∠ C = ∠ A.

The △ A B D is also isosc., and its ∠s are equal to the ∠s of △ A C B.

Hence △s A B D, A C B, are similar, and (by 4, VI.) the sides about the equal ∠s are proportional; *i.e.* A C : A B :: A B : A D; and A B = B D = C D; ∴ A C : C D :: C D : A D.

Thus when C A, the rad. of a circle, is divided, at D, into extreme and mean ratio (11, II.) the greater segment, C D, will be the side of the decagon inscribed in that circle.

Hence, PROB. 1.—*To find the side of an inscribed decagon, when the rad.,* C A, *of a circle is given.*

By 11, II., divide the rad. C A into extreme and mean ratio in D, and the greater segment C D equals the side of the decagon required.

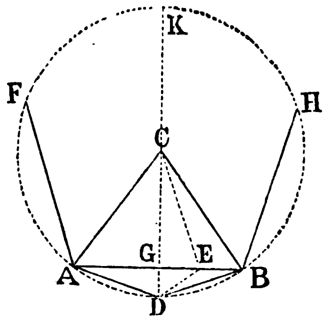
PROB. 2.—*To find the radius of the circumscribing circle when the side* A B, *of the regular decagon is given.*

By 11, II., produce B A, so that the rectangle contained by the whole line produced BE, and the part produced, A E, *i. e.* B E · A E = A B²; the whole line thus produced, B E, is the radius of the circumscribing circle.

SECOND. Of the *Pentagon.*

By joining the alternate angular points of the decagon a corresponding Pentagon will be drawn.

II. To demonstrate, *that the square on the side* A B *of a reg. pentagon inscribed in a circle equals the sum of the squares of the rad.* A C, *and of the side* A D, *of the inscribed decagon; i.e.* A B² = A C² + A D².



Let the sides of the inscribed decagon be A D, D B; join A, B; then A B is the side of the inscribed pent.

Bis. ∠ B C D by C E, and join E D.

Then ∵ D C, C E = B C, C E, and ∠ D C E = ∠ B C E;

∴ (4, I.) D E = E B, and △ C D E = △ C B E.

And in △s B E D, A D B, ∵ ∠ E B D is com. ∠ E D B = ∠ B A D, and ∠ B E D = ∠ A D B,

∴ the △s B E D, A D B, are similar.

Hence (4, VI.) A B : B D :: B D : B E, or $BD^2 = AB \cdot BE$ (1).

Again, ∵ the rad. C A bisects the ∠ B A F, of a regular figure, ∴ ∠ C A E = ½ ∠ B A F of the inscribed pentagon;

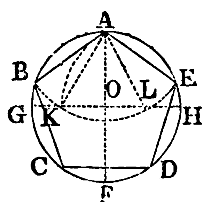
And ∵ the *five* angles together = 6 rt. ∠s, ∴ the half of one of the five, namely ∠ C A E = 3-5ths of a rt. ∠.

But, ∵ ∠ A C D = 2-5ths of a rt. ∠, and ∠ D C E = 1-5th of a rt. ∠, ∴ ∠ A C E = 3-5ths of a rt. ∠.

Thus, ∠ C A E = ∠ A C E, and △ A E C is isosceles and similar to △ A C B.

Hence (4, VI.) A B : A C :: A C : A E; or $AC^2 = AB \cdot AE$ (2).

Adding Equations (1) and (2); $BD^2 + AC^2 = AB \cdot BE + AB \cdot AE =$ (2, II.) AB^2.



Cor.—Hence a *Practical Method* for inscribing a pentagon in a given circle A D B E.

Draw diam. G H at rt. ∠s to diam A F;
bisect the rad. O H in L, and join L A;
make L K = L A, and join K A;
K A is the length of the side of the pentagon required.

N.B.—The rad. O G is cut in extreme and mean ratio in the point K; and O K is the length of the side of an inscribed decagon.

Prop. 15.—Prob.

To inscribe an equilateral and equiangular hexagon in a given circle.

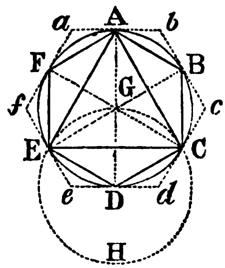
Con. 1, III. Psts. 3. 1. 2.

DEM. Def. 15, I. Ax. 1. 32, I. 26, III. 29, III. Ax. 2. 27, III. Def. 3, IV.

Cor. 5, 1. Every equil. △ is also equiangular.

13, I. The ∠s which one st. line makes with another upon one side of it are either rt. ∠s, or together equal to two rt. ∠s.

15, I. If two st. lines cut one another, the opp. or vert. ∠s shall be equal.



E. 1	Datum.	Let ACE be a given circle;
2	Quæs.	to inscribe in it a reg. hexagon.
C. 1	1, III.	Find G the cen. of ⊙ ACE, and draw the diam. AD;
2	Pst. 3.	from D, with DG, desc. ⊙ ECH;
3	Psts. 1 & 2.	join EG, CG, and prod. them to B, F;
4	Pst. 1.	join also AB, BC, CD, DE, EF, FA;
5	Sol.	the fig. ABCDEF is the hexagon required.
D. 1	C. 1 & 2.	∵ G is cen. of ⊙ ACE, and D of ⊙ ECH;
2	Def. 15, I. Ax. 1.	∴ GE = GD, and DE = GD; ∴ GE = ED;
3	D. 2. Cor. 5, I.	and ∵ △ EGD is equil.; ∴ ∠ EGD = ∠ GDE = ∠ DEG;
4	32, I.	but the 3 ∠s of a △ = 2 rt. ∠s; ∴ ∠ EGD = $\frac{1}{3}$ of 2 rt. ∠s.
5	*Sim.*	So ∠ DGC = $\frac{1}{3}$ of 2 rt. ∠s;
6	13, I.	and ∵ GC with EB makes ∠s EGC, CGB, = 2 rt. ∠s;
7	Conc.	∴ remg. ∠ CGB = $\frac{1}{3}$ of 2 rt. ∠s;
8	D. 4, 5, 7.	∴ ∠ EGD = ∠ DGC = ∠ CGB;
9	15, I.	to these ∠s are equal the vert. ∠s BGA, AGF, FGE;
10	D. 8. 9.	∴ the *six* ∠s are eq.; EGD = DGC = CGB = BGA = AGF = FGE.
11	26, III.	But eq. ∠s stand on eq. arcs,
12	Conc.	∴ the *six* arcs are eq.; AB = BC = CD = DE = EF = FA;
13	29, III.	and eq. arcs have eq. chords;
14	Con	∴ the *six* chords are eq.; and ABCDEF is equil.
15	D 13. *Add.*	Again, ∵ arc AF = arc ED, to each add arc ABCD;

16	Ax. 2.	∴ the whole arc F A B C D = the whole arc E D C B A.
17	C.	Now ∠ F E D stands on arc F A B C D, and ∠ A F E = on arc F D C B A;
18	27, III.	∴ ∠ A F E = ∠ F E D;
19	*Sim.*	So ∠s A B C, B C D, C D E each = ∠ A F E = ∠ F E D;
20	D. 14 and 10.	∴ the hexagon is eq. lat. and eq. angular,
21	Def. 3, IV.	and it is inscribed in ⊙ A C E.

Q. E. F.

Cor. 1.—*The side of a regular hexagon inscribed in a circle is equal to the radius, or semi-diameter, of the circle*; or, in other words, *the chord of* 60° *is equal to the radius.*

D.1	15, IV. D. 2.	∵ G E = E D; E D being the side of the hexagon, and G E the rad.;
2	Conc.	∴ E D the side, or chord of 60° = G E the radius.

Cor. 2.—*An equilateral triangle would be inscribed by joining the points* A, E, *and* C, *alternate points in the hexagon.*

Cor. 3.—*Every equil. figure inscribed in a* ⊙ *is also equiangular; for* ∵ *its* ∠ *s are contained by the chords of equal arcs, and* (28, III.) *stand on equal arcs,* ∴ *its* ∠ *s are all equal, each to each.*

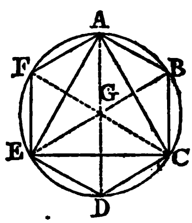
Sch.—1. The opposite sides, A B and D E, or B C and E F, or C D and F A, of a hexagon are parallel; for, ∵ A D meeting A B and D E makes the alt. ∠ B A D = the alt. ∠ A D E, ∴ (27, I.) A B ∥ D E.

2. If, through the angular points of the inscribed hexagon, A, B, C, D, E, F, tangents to the circle be drawn, *ab*, *bc*, *cd*, *de*, *ef*, *fa*, these will form a regular hexagon circumscribing the circle.

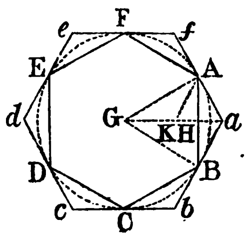
The Proof is the same as in Prop. 12 and 13, IV.

Use and App.—1. *On a given line,* A B, *to describe a regular hexagon.*

C.1	1, I.	On A B draw an eq. lat. △ A G B;
2	Pst. 3.	from G, with G A, or G B, desc. a ⊙ A C E;
3	Sol.	then A B will exactly divide the ⊙ce into *six* equal parts.
D.1	C. 1.	∵ G A, G B = G C, G B, and ∠ A G B = ∠ B G C;
2	4, I. & *Sim.*	∴ A B = B C; and B C = C D = D E = E F = F A.
3	Conc.	∴ fig. A B C D E F is a regular hexagon.



2. *The inscribed hexagon,* A B C D E F, *in a circle is three fourths of the area of the circumscribed hexagon, a b c d e f.*



E.1	Dat.	Let circumsc. hex., *a b c d e f*, touch the circle A C E, at the $\angle$s A, B, C, D, E, F, of the inscribed hex. A B C D E F.
C.1	Pst. 1.	Join G, and any angular · as *a* of the circumscribing hex.;
2	Pst. 1.	join also G A, G B, adjacent points of contact ;
3	10, I.	Bis. G *a* in K, and join A K ;
4	Conc.	then $\triangle$ AGH $= \frac{3}{4}$ of $\triangle$ A G *a*.
D.1	8, I.	$\because \triangle$ G A *a* $= \triangle$ G B *a*; $\therefore \angle$ A G *a* $= \angle$ B G *a* ;
2	4, I. Def. 10, I.	and $\because \triangle$ GAH $= \triangle$ GBH ; $\therefore \angle$s GHA, GHB, are rt. $\angle$s.
3	Cor. 1, 15, IV.	But G *a* = G *f* = *f a* ; and G A is perp. to *f a*,
4	18, III.	$\therefore$ also G A bisects *f a*.
5	C. 2.	and $\because$ A K bisects G *a*, A K = K *a* = A *a* ;
6	Conc.	$\therefore \triangle$ A K *a* is. eq. lat. ; and K *a* bisected by A H ;
7	C. 2. 38, I.	but $\because$ G K = K *a*, $\therefore \triangle$ G A K $= \triangle$ *a* A K ;
8	D. 6. 38, I.	and $\because$ K H = H *a*, $\therefore \triangle$ K A H $= \triangle$ H A *a* ;
9	Remk.	thus $\triangle$ K A *a* $= \frac{1}{2} \triangle$ G A *a*, and $\triangle$ H A *a* $= \frac{1}{4}$ G A *a* ;
10	Conc.	$\therefore \triangle$ G A H $= \frac{3}{4} \triangle$ G A *a*.
11	*Sim.*	So, by drawing lines from G to the angr. · s of the inscribed and circumscribed hexagons, each $\triangle$ of the inscrd. hex. $= \frac{3}{4}$ of each $\triangle$ of the circumscrd. hex.
12	Conc.	$\therefore$ the inscr. hex. A B C D E F $= \frac{3}{4}$ of the c. scr. hex. *a b c d e f*.

Q. E. D.

3. The area of a reg. hex. A B C D E F is *six times* the area of the eq. lat. $\triangle$ A G B described on the same st. line, A B.

4. Because the side of a reg. hexagon is equal to the chord of 60°; and the chord of 60° equals the rad.; $\therefore \frac{1}{2}$ *rad. = sine of* 30°.

5. The inscribed hexagon and the successive bisections of its arcs, have been employed as the groundwork for finding *the approximate ratio of the circumference of a circle to its diameter.* By forming Polygons of any number of sides that are successive bisections of the original 6, as 12, 24, 48, &c., we obtain the *apothem*; *i. e.*, the perpendicular from the centre on one of the sides, which continually approaches the radius in length ; and from a polygon of 1536 sides we deduce the approximate values of the circumference and radius to be 6·283185 and 1.

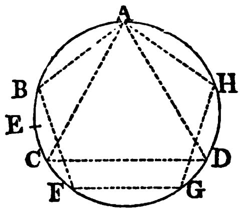
When a *square* is taken as the groundwork of the process, and the *apothems* and sides of successively inscribed polygons of 8, 16, 32, &c., sides are employed ; on arriving at the polygon with 32768 sides, the Areas of the inscribed and circumscribed Polygons agree to the seventh place of decimals, 3·1415926, and since the area of the circle is intermediate, this value, 3·1415926, as far as it goes, must also be the area of the corresponding circle. Now, by Use 4, Pr. 41, Bk. I., the *Area of a circle is equal to the rectangle under its circumference and semi-radius, or under its radius and semi-circumference;* therefore the radius being 1, the semi-circumference is 3·1415926 ; and since the diameter is twice the radius, 3·1415926 to 1 is also the Ratio of the circumference to the diameter.—*See Chambers's Euclid, pp.* 202-221, *and Penny Cyclopædia, Vol. XIX., pp.* 186, 187.

Prop. 16.—Prob.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Con. 1, I. 2, IV. Cor. 6, I. 11, IV.
Cor. 5, I. Every eq. lat. △ is also eq. ang.
28, III. To bisect a given arc of a circle.

Dem. 28, III. 1, IV. 11, IV.

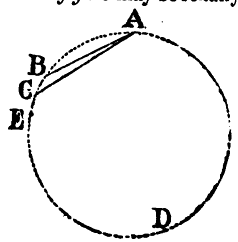


E.1	Dat.	Let A B F D be the given ⊙;
2	Quæs.	in that ⊙ to inscr. a reg. quindecagon.
C.1	1,I.Cor.5,I.	Desc. an eq. ang. and eq. lat. △;
2	2, IV.	and in ⊙ A B F D inscr. △ A C D eq. ang. to it;
3	Cor. 6, I.	then △ A C D will be eq. lat.
4	11, V.	Also insc. in ⊙ A B F D a reg. pent. ABFGH;
5	Sup.	and let one angr. • A be com. to the △ and the pent.
6	30, III.	Bis arc B C in E, and join E C, E B;
7	Sol.	then E C is a side of the quindecagon, to be set round the ⊙ A B F D.
D.1	28, III.	∵ equal chords cut off equal arcs,
2	Conc. 1.	∴ arc A C = arc C D = arc D A;
3	Conc 2.	and ∴ arc A C = $\frac{1}{3}$ ⊙ce.
4	*Sim.*	So also arc A B = $\frac{1}{5}$ ⊙ce.
5	D. 3. 4.	Thus of the 15 eq. arcs in the ⊙ce, arc A C contains 5 and arc A B, 3;
6	Ax. 3.	∴ arc B C, equalling arc A C *minus* arc A B, contains 2 of the 15;
7	C. 6.	∴ arc B E = arc E C = $\frac{1}{15}$ of the ⊙ce.
8	1. IV.	Hence, if st. lines each = chord B E, be placed in succession round the ⊙ A B F D;
9	Conc.	then a reg. quindecagon will be inscribed in the given ⊙.
10	11, IV.	The proof will be similar to that for the pentagon.

COR.—The only reg. st. lined figures which can be placed, side by side, so as to make a continuous plane surface, are the equilateral triangle, the square, and the hexagon.—*See Use* 3. *Pr.* 15, *Bk. I.*

SCH.—1. As in the Pentagon, 12, IV., if through the angular points B, E, C, F, &c., of the inscribed quindecagon, tangents to the circle be drawn, *a regular quindecagon will be circumscribed:* and, as in the Pentagon, 13 and 14, IV., a circle may be inscribed in a given quindecagon, or circumscribed about it.

2. The arc subtending a side of *a regular thirty-sided figure* may be readily found, by placing in a circle, ABCD, from the same point A, the line AB = the side of an inscribed hexagon, and AC = the side of an inscribed pentagon;



then, the arc BC = the arc of a reg. thirty-sided figure;

for 1-5th of a ⊙ *minus* 1-6th of a ⊙ = 1-30th of the same circle.

Also *twice* BC = BE = 1-15th the arc of a regular quindecagon.

USE.—This proposition opens the way for the *construction of other polygons*; for, if we obtain the common measures of 360°, as 2, 3, 4, 5, 6, 8, 9, 10, &c., and divide 360° by any two successive common measures, the difference of their quotients will give an arc of the circle, by which the circle will be exactly measured; consequently, a regular polygon may be inscribed, the sides of which are equal in number to 360 divided by the difference of the quotients arising from the division of 360° by the two successive common measures:

Thus, 1-9th of 360 = 40; 1-10th of 360 = 36; and 40 − 36 = 4;

and $\frac{360}{4} = 90$, the number of parts into which the circle would be divided, and the number of the sides of the polygon.

Again, 1-36th of 360 = 10; 1-40th of 360 = 9; and 10 − 9 = 1, the difference of the arcs; and 1 will give a polygon of 360 sides.

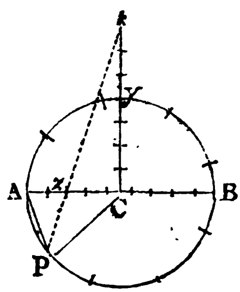
OBSERVATIONS ON POLYGONS.

Obs. I.—The only known ways of dividing a circle geometrically are—

1st, into	3, 6, 12, 24, &c., parts	—by Prop.	15, IV.,	& Prop.	9, I.
2nd, ,,	4, 8, 16, 32, &c. ,,	,,	6, IV.,	,,	9, I.
3rd, ,,	5, 10, 20, 40, &c. ,,	,,	11, IV.,	,,	9, I.
4th, ,,	15, 30, 60, 120, &c. ,,	,,	16, IV.,	,,	9, I.

Obs. II.—Consequently there are many polygons which cannot be inscribed in a circle, except by mechanical contrivances; but the following methods *approximate* so near to the truth, that they may be adopted without sensible error.

1. *In a given circle, as* A B P, *to inscribe any regular rt. lined figure; or to divide the circumference of a circle into any assigned number of equal parts.*



C.1	U. & A. 34, I.	Divide the diam. A B, into the assigned number of eq. parts, as 9;
2	11, I.	from cen. C. raise a perp. C k;
3	U. & A. 34, I.	div. rad. C y into 4 eq. parts;
4	3, I.	and set off 3 of those parts from y to k;
5	Pst. 1.	join k and z, the second of the divisions from A;
6	Pst. 2, & 1.	prod. k z to ⊙ce, P; and join P C, A P;
7	Sol.	the line A P = the side of required figure;
8	Remk.	and A P set round the ⊙ will divide it into the assigned number of parts.

N.B. It is usual to denote the *circumference*, measuring four rt. angles, by 2π; the *semi-circumference*, by π; and the *number of sides* of the figure, or equal parts of the circle, by n.

Analysis.—The $\angle$ A C P is given $= \frac{2\pi}{n}$;

also $\angle$ C A P or $\angle$ C P A $= 90° - \frac{2\pi}{2n}$;

A C $= \frac{n}{2}$ being given, C z $= \left(\frac{n}{2} - 2\right)$ is given;

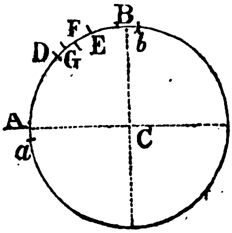
$\therefore$ P z can be found; as also $\angle$ A z P, or its equal $\angle$ C z k; hence the compl. of the $\angle$ C z k, namely $\angle$ C k z, can be found.
And C z being given, C k can be found; and hence k y can be found;
and $k\,y = \frac{3}{4}$ C y nearly.—*See Treatise on Mensuration, Irish National Schools*, p. 19. *Demonstrations*, p. 53.

2. Though no exact geometrical rule has been discovered for inscribing many figures, as the heptagon, enneagon, or nonagon, hendecagon, &c., figures of 7, 9, 11, 13, &c., sides,—yet a high degree of accuracy may be obtained by the following *approximative process*, the real errors of which are far less than those which the imperfections of our best instruments entail in all geometrical constructions.

By 9, I. continue the series of bisections of the circle and of its arcs 2, 4, 8, 16, 32, 64, &c., until a number be found greater or less by *one* than a multiple of the number of sides in the required polygon. Of the equal parts thus found, take as many as constitute a multiple part of the required polygon, and let one more of the equal parts obtained by successive bisections be also bisected successively, until its parts are one more than the number of the sides of the polygon;—then *the first multiple part* plus *the second multiple part* will, with sufficient accuracy, be equal to the side of the required polygon.

Thus, *for the Heptagon:*

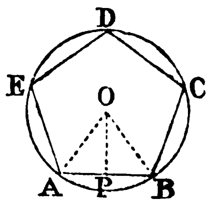
The continued bisection of the ⊙ce, until it is divided into 64 equal parts, gives a number greater by *one* than 9 × 7, or 63, a multiple of 7.



Now by bisecting the quadrantal arc A B in D, arc D B in E, arc D E in F, and arc D F in G, (A G being equal to 9 out of the 64 parts), the arc A G will be less than a *seventh* part of the circumference by a *seventh* part of one of them, D G.

But the arc D G being small, a *seventh* part of its *chord* may, without any considerable error, be assumed for the *seventh* part of the *arc* itself, being somewhat *less* than the latter;—and if the chord of A *a* be taken equal to this approximate seventh part, the error of assuming for it the arc A *a*, which is somewhat *greater* than its chord, will be still less,—so that arc G *a* will be equal, very nearly, to one-seventh of the circumference,—and the chord of G *a*, very nearly equal to the side of a regular heptagon inscribed in a circle.—*Geometry, Plane, Solid and Spherical*, p. 121.

Obs. III.—In the construction of polygons it is useful *to ascertain* the magnitude either of *the angle*, A O B at the centre, or of the *angle* A B C, formed by two adjacent sides.



As before, in the formulas, let π denote 2 rt. $\angle$s, or the semi-circumference; 2π, 4 rt. $\angle$s, or the whole circumference; n, the number of sides; and θ the magnitude of one angle, either at the centre, or at the concave boundary.

1*st*. *To find the magnitude of an $\angle$ at the centre, as $\angle$ A O B of a polygon.*

$n\,\theta =$ sum of all the $\angle$s at the centre, or $2\,\pi = 4$ rt. $\angle$s, (Cor. 2. 15, I.)

$\therefore \dfrac{2\,\pi}{n} = \theta$, the magnitude of one $\angle$, as $\angle$ A O B at the centre.

2*nd*. *To find the magnitude of an angle, as $\angle$ A B C, formed by two adj. sides of a polygon.*

$n\,\theta =$ the sum of the magnitudes of all the interior $\angle$s;
But, Cor. 1. 32, I., $n\,\theta + 2\,\pi = n\,\pi$;
and by trans. $n\,\theta = n\,\pi - 2\,\pi = (n - 2).\ \pi$.

$\therefore\ \theta = \dfrac{(n-2)}{n} .\ \pi =$ the mag. of one $\angle$, as $\angle$ A B C, formed by two adj. sides.

Note.—For n substitute the number of sides, 3, 4, 5, 6, &c.; and for $2\,\pi$, 360°, and the magnitude of the angles is found in numbers.

Obs. IV.—*On a given rt. line,* AB, *to construct a reg. polygon.*

1. By dividing $2\pi = 360°$ by n the number of sides, we obtain the arc AB, or the measure of the $\angle$ at the cen. $\angle$ AOB; and $\triangle$ AOB being isosc., $\angle$s OAB, OBA are equal.

Now $\dfrac{180° - \angle \text{AOB}}{2} = \angle \text{OAB} = \angle \text{OBA};$

and $\because \angle \text{ABO} = \frac{1}{2} \angle \text{ABC}, \therefore 180° - \angle \text{AOB} = \angle \text{ABC},$ one of the angles of the polygon to which all the others are equal.

Hence the Rule, or Formula;

$\pi - \dfrac{2\pi}{n} = \angle$ ABC, one of the eq. $\angle$s of the polygon;

at A and B make $\angle$s OAB, OBA each $= \dfrac{\angle \text{ABC}}{2}$;

then from O, with OA, or OB, desc. a $\odot$;

and in this circle place the st. line AB continually;
a polygon of the assigned number of sides will be drawn.

2. By means of Trigonometry the radius of the circumscribing circle is calculated, assuming the side of the polygon to be 1, and the following Table constructed.

No. of Sides.	Name of Polygon.	Radius of Circumscribing Circle.	Angle OAB or OBA.
3	Trigon	·5773503	30°
4	Tetragon	·7071068	45°
5	Pentagon	·8506508	54°
6	Hexagon	1·Side = radius	60°
7	Heptagon	1·1523825	$64\frac{2}{7}$°
8	Octagon	1·3065630	$67\frac{1}{2}$°
9	Nonagon	1·4619022	70°
10	Decagon	1·6186340	72°
11	Undecagon	1·7747329	$73\frac{\ }{11}$°
12	Dodecagon	1·9318516	75°

Then, the units of length in AB $\times$ tabular rad. $=$ units of length in OA or OB, the rad. of the actual circle, in which the st. line AB is to be placed, step by step, so that the polygon may be formed.

OBS. V.—To calculate numerically the *area of any regular polygon*, it is sufficient to find the area of one of the triangles into which the polygon may be divided, and to multiply that area by the number of triangles;

Or, *given a side, as* AB, *and* OP *the perpendicular, from* O *the centre, on* AB;

then, $\frac{n\,AB}{2} \times OP =$ the area of the polygon.

To facilitate the calculation *when only a side, as* AB, *is given*, the following Table has been formed on trigonometrical principles—the side of the polygon being 1.

No. of sides.	Radius of Inscribed Circle.	Area of Polygon, or Multiplier.	Trigonometrical Expression.
3	0·2886751	0·4330127	$= \frac{3}{4}$ tang. $30° = \frac{1}{4}\sqrt{3}$.
4	0·5	1·	$= \frac{4}{4}$ tang. $45° = 1 \times 1$.
5	0·6881910	1·7204774	$= \frac{5}{4}$ tang. $54° = \frac{5}{4}\sqrt{(1 + \frac{2}{5}\sqrt{5})}$.
6	0·8660254	2·5980762	$= \frac{6}{4}$ tang. $60° = \frac{6}{4}\sqrt{3}$.
7	1·0382617	3·6339124	$= \frac{7}{4}$ tang. $64\frac{2}{7}°$.
8	1·2071068	4·8284271	$= \frac{8}{4}$ tang. $67\frac{1}{2}° = 2 \times (1 + \sqrt{2})$.
9	1·3737387	6·1818242	$= \frac{9}{4}$ tang. $70°$
10	1·5388418	7·6942088	$= \frac{10}{4}$ tang. $72° = \frac{5}{2}\sqrt{(5 + 2\sqrt{5})}$.
11	1·7028437	9·3656404	$= \frac{11}{4}$ tang. $73\frac{7}{11}°$
12	1·8660254	11·1961524	$= \frac{12}{4}$ tang. $75° = 3 \times (2 + \sqrt{3})$.

By using the Table we find,

$AB^2 \times$ tabular area $=$ area of the reg. polygon;

or, $AB \times$ rad. of inscr. $\odot \times \frac{n\,AB}{2} =$ area.

OBS. VI.—DODSON, in his "Calculator," supplies the following Tables for the calculation and construction of any reg. polygon having not more than 12 sides.

1° *When the length of the side* = 1.

Sides.	Radius of Circumscribed Circle.	Radius of Inscribed.	Area.
3	0·5773503	0·2886751	0·4330127
4	0·7071068	0·5	1·
5	0·8506508	0·6881910	1·7204774
6	1·	0·8660254	2·5980762
7	1·1523825	1·0382617	3·6339124
8	1·3065630	1·2071068	4·8284271
9	1·4619022	1·3737387	6·1818242
10	1·6180340	1·5388418	7·6942088
11	1·7747329	1·7028437	9·3656404
12	1·9318516	1·8660254	11·1961524

2° *When radius of circumscribed circle* = 1.

Sides.	Length of Side.	Radius of Inscribed Circle	Area.
3	1·7320508	0·5	1·2990381
4	1·4142136	0·7071068	2·
5	1·1755705	0·8090170	2·3776412
6	1·	0·8660254	2·5980762
7	0·8677674	0·9009689	2·7364102
8	0·7653668	0·9238795	2·8284271
9	0·6840403	0·9396926	2·8925437
10	0·6180340	0·9510565	2·9389263
11	0·5634651	0·9594931	2·9735250
12	0·5176381	0·9659259	3·

3° *When radius of inscribed circle* = 1.

Sides.	Length of Side.	Rad. of Circumscribed Circle.	Area.
3	3·4641016	2·	5·1961524
4	2·	1·4142136	4·
5	1·4530851	1·2360680	3·6327128
6	1·1547005	1·1547005	3·4641016
7	0·9631491	1·1099160	3·3710222
8	0·8284271	1·0823919	3·3137084
9	0·7279405	1·0641776	3·2757315
10	0·6498394	1·0514622	3·2491970
11	0·5872521	1·0422172	3·2298913
12	0·5358984	1·0352760	3·2153904

4° *When the area* = 1.

Sides.	Length of Side.	Rad. of Circumscribed Circle.	Radius of Inscribed Circle.
3	1·5196716	0·8773827	0·4386912
4	1·	0·7071068	0·5
5	0·7623870	0·6485251	0·5246678
6	0·6204033	0·6204033	0·5372849
7	0·5245813	0·6045183	0·5446520
8	0·4550899	0·5946034	0·5493420
9	0·4201996	0·5879764	0·5525172
10	0·3605106	0·5833184	0·5547687
11	0·3267617	5·5799148	0·5564242
12	0·2988585	0·5773503	0·5576775

Use and App.—By the help of these tables, and of the compasses, and a scale of equal parts, the construction of any regular st. lined figure is reduced to a simple calculation, to the drawing of a circle, and to the setting off of equal chords on that circle.

Ex.—To construct a reg. heptagon with an area 144 times greater than the square on any one division of the scale of eq. parts.

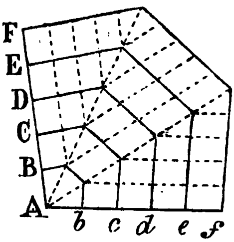
Increase the side and rad., in Table 4. in the proportion of $\sqrt{144}$ to $\sqrt{1}$ or of 12 to 1; then

$\cdot 5245813 \times 12 = 6\cdot 2949756 =$ length of a side of the hept.
$\cdot 6045183 \times 12 = 7\cdot 2542196 =$,, of rad. of circumscribed ⊙,
$\cdot 5446520 \times 12 = 6\cdot 5358240 =$,, of rad. of inscribed ⊙,

From the same centre with the two radii, draw two circles; the side found above will measure the larger circle into seven steps; and chords, joining the ·s in which the ⊙ce is cut, will touch the inner circle.

Thus, a reg. heptagon will be inscribed, or circumscribed.

Obs. VII.—In Arithmetic *polygonal numbers* are such as are the sums of a series of numbers beginning with *unity* and so increasing as to be representative of the figure of a polygon. These numbers are subdivided into triangular, quadrangular, pentagonal, &c., and may be explained by taking a pentagonal number.



1. Construct a set of pentagons A *c* C, A *d* D &c., double, treble, &c. of A *b* B, in lineal dimension;

Divide the sides of each pent. into parts each = corresponding side A *b* B;

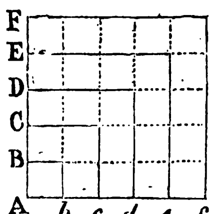
Then, beginning with A, *one* point, and taking in all the points of pent. A *b* B, we have 1 + 4 = 5 points;

Add all the ·s of pent. A *c* C that are not in pent. A *b* B, and we have 1 + 4 + 7 = 12 points;

So for pent. A *d* D, 1 + 4 + 7 + 10 = 22 points;

For pent. A *e* E, 1 + 4 + 7 + 10 + 13 = 35 points;

And pent. A *f* F, 1 + 4 + 7 + 10 + 13 + 16 = 51 points &c.



This series 1, 5, 12, 22, 35, 51, &c., is a series of pentagonal numbers, in a way similar to that in which 1, 4, 9, &c., are a series of square numbers. By aid of the Square on A *f*, the series of square numbers will be readily formed.

2. *To find the numbers which bear the name of an* n*-sided figure.*

The mth number of the n-sided fig. $= 1 + nm \cdot \frac{m-1}{2} - (m-1)^2$

Form a series of terms beginning with 1, and, by a com. difference $= n - 2$, increasing in Arithmetical progression;

then the sums of the terms of the arithmetical series form the series of polygonal numbers.

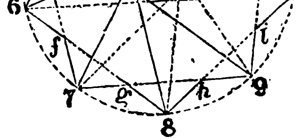
Thus, for Decagonal numbers, in which $n - 2 = 8$;
the series 1, 9, 17, 25, 33, 41, 49 &c. in arithmetical progression;
gives 1, 10, 27, 52, 85, 126, 175 &c. Decagonal numbers.

Some of the polygonal numbers are,

Triangular, 1, 3, 6, 10, 15, 21, &c.
Quadrangular, 1, 4, 9, 16, 25, 36, &c.
Pentagonal, 1, 5, 12, 22, 35, 51, &c.
Hexagonal, 1, 6, 15, 28, 45, 66, &c.
Heptagonal, 1, 7, 18, 34, 55, 81, &c.

For a further account, *see Penny Cyclopædia, Vol.* 16, *p* 364.

Obs. VIII. As it has been observed, Euclid treats only of *regular convex polygons;* though, according to his definition of a reg. st.-lined figure, they are both equilateral and equiangular, he does not name *Star-shaped polygons*, *i.e.* regular polygons with *re-entrant* angles, as *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*. Figures of this kind are described, by first constructing any regular convex polygon, as might be at the points 1, 2, 3, 4, 5, 6, 7, 8, 9; and by then drawing successive diagonals, as 1—3, 2—4, 3—5, &c., so as to cut off that number of sides which is *prime* to the number of sides of the assumed convex polygon; or what leads to the same results, so as to cut off a number of sides which neither measures the number in the given convex polygon, nor is measured by any of its common measures.



To 5 the no. of sides in a pentagon, 2 and 3 are the prime numbers;
7 „ heptagon, 4 and 3, and 5 and 2 „
9 „ nonagon, 2 and 7, and 4 and 5 „
12 „ dodecagon, 5 and 7 „
13 „ tridecagon 2 and 11, 3 and 10, 4 and 9, 5 and 8, 6 and 7;

And diagonals cutting off successively in a

Pentagon, 2 and 3 sides,	will form ...	1	star-shaped polygon;
Heptagon, 4 and 3, or 5 and 2	will form ...	2	„ polygons;
Nonagon, 2 and 7, or 4 and 5	will form . .	2	„ polygons;
Dodecagon, 5 and 7,	will form ...	1	„ polygon;
Tridecagon, 2 and 11, 3 and 10, 4 and 9, 5 and 8, or 6 and 7 sides	will form ...	5	„ polygons.

A similar process will form other star-shaped polygons.

N.B.—The figure as C at the centre of a star-shaped polygon, formed by the intersection of lines from the angular points, 1, 2, 3, &c., is also a regular convex polygon, with the same number of sides as there are re-entrant angles to the star-shaped polygon.

REMARKS ON BOOK IV.

1. The Problems of which the Fourth Book of Euclid is entirely composed require little classification; they may be brought under the general heads;—

1st, Of rt.-lined regular figures inscribed in circles; Pr. 2, 6, 11, 15 and 16;

2nd, Of rt.-lined regular figures, circumscribed about circles; Pr. 3, 7, 12;

3rd, Of circles inscribed in regular rt.-lined figures; Pr. 4, 8, 13;

4th, Of circles circumscribed about rt.-lined figues; Pr. 5, 9, 14;

5th, Of constructing an isosceles triangle, having each angle at the base double of the vertical angle, Pr. 10.

2. The Use and Application of the Propositions of this Book have been given so much *in extenso*, that it would be superfluous to add to these remarks, except again to challenge attention to the value of theoretical reasoning as the guide in Geometry to most important Practical Results.

GRADATIONS IN EUCLID.

BOOK V.

THE THEORY OF PROPORTION, OR OF THE COMPARATIVE MAGNITUDES OF PLANE FIGURES.

This Book is entirely independent of the four books which preceded it. In the main they relate to the Properties of Figures on a plane surface, but the fifth book introduces Properties of a more general kind, and though restricted by Euclid to the comparative magnitudes of right lines, really extends to all kinds of magnitude. It is no longer absolute equality, or inequality, which we have to consider, but the RATIO, or mode of estimating the relative lengths of lines and the magnitudes of figures, and the *proportion*, or setting forth of those relative lengths or magnitudes.

As in the Second Book material assistance for illustrating the properties of Rectangles, was derived from Algebra and Arithmetic, so in this book similar help will be obtained from the same sources. Indeed very many of the terms employed will be already familiar from their use in Arithmetic; and it will be seen that the estimating of Ratios, and the setting forth of proportions, rest so entirely on a numerical basis, that to a very high degree the Fifth Book, or the Theory of Proportion, is an application of Numbers to the purposes of Plane Geometry.

For this reason it will be of advantage to the Learner to be presented with a brief view of the Principles on which are established the Properties of Proportional Numbers and Quantities. If, however, he has already mastered the subject, he may pass over the next few pages, and at once enter on the Theory of Geometrical Proportion.

SOME PROPERTIES OF PROPORTIONAL NUMBERS,

INTRODUCTORY TO EUCLID'S THEORY OF GEOMETRICAL PROPORTION.

We may compare two numbers together, either by their *difference*, or by their *quotient*, *i.e.* by the number of times which the greater contains the less, or the less measures the greater. When we say $12 - 9 = 3$, we compare 12 and 9 by their difference; but when we say 12 *contains* 9, *one* and *one quarter* times, we compare them by a division of 12 into 9 and a part of 9.

The proportion which one number or quantity bears to another is often called its *Ratio*; the ratio, measured by the *difference* is named an *arithmetical* ratio,—that measured by the quotient,—a *geometrical* ratio.

Proportion is applied, *either* to *an identity of difference* between three or more numbers, as 12, 9, 6, where the common difference or the Arithmetical ratio is 3; *or*, to *identity of relative magnitudes*, as $12 : 9 :: 8 : 6$,—where 12 contains 9 just as often as 8 contains 6,—the common quotient or ratio being $1\frac{1}{3}$. When the differences are identical, the numbers are in Arithmetical Proportion; when the contents of each pair of terms are identical, the numbers are in Geometrical Proportion. The term proportion, taken by itself, is usually restricted to numbers in geometrical proportion; and of these we have now to treat.

Identity in the quotients of successive pairs of numbers constitutes Proportion. Take for an example, $15 : 5 :: 36 : 12$; the quotient obtained on dividing 15 by 5 is the same as that obtained by dividing 36 by 12; and these four numbers, 15, 5, 36, and 12,—or any other four numbers fulfilling the condition of equality of quotients in each successive pair, form a Proportion, or set of Proportionals.

The *extremes* are the first and last terms in the series; the terms placed between the first and last terms, the *means*; the *antecedent* is the first term of a ratio; the *consequent*, the second term.

The *ratio* may be expressed, either by a whole number or by a fraction; thus in the proportionals, $18 : 6 :: 24 : 8$, the constant ratio is 3; in the proportionals $12 : 9 :: 36 : 27$ that ratio is $\frac{4}{3}$ or $1\frac{1}{3}$.

If we take two sets of quantities in direct proportion, $a : b :: c : d$ (which may also be written in a fractional form $\frac{a}{b} = \frac{c}{d}$) we can readily exhibit various rules that are employed in modifying a Proportion: they are all dependent on the principle that *resulting equations are equally true whenever the thing which is done on one side of an equation is also done on the other side.* Take as an example $\frac{a}{b} = \frac{c}{d}$.

RULE 1. Multiply each side of the equation by $b \times d$; we obtain $\frac{abd}{b} = \frac{cbd}{d}$, or $ad = cb$; $\therefore$ the product of the extremes = the product of the means; and *conversely*, if the product of the extremes = the product of the means, the quantities are proportional; for, dividing each side by $b\,d$, we have $\frac{ad}{bd} = \frac{cb}{bd}$, or $\frac{a}{b} = \frac{c}{d}$, i.e., $a : b$ & $c : d$ are proportionals.

RULE 2. Multiply each side by $\frac{b}{c}$, the sides become $\frac{ab}{bc} = \frac{cb}{dc}$ or $\frac{a}{c} = \frac{b}{d}$, i. e. $a : c = b : d$.

RULE 3. Add *unity* to each side, and the equation is $\frac{a}{b} + 1 = \frac{c}{d} + 1$; or reducing the mixed quantity, $\frac{a+b}{b} = \frac{c+d}{d}$ *i.e.*, $a + b : b = c + d : d$.

RULE 4. Subtract *unity* from each side, we have the equation $\frac{a}{b} - 1 = \frac{c}{d} - 1$; or $\frac{a-b}{b} = \frac{c-d}{d}$ i.e., $a - b : b = c - d : d$.

RULE 5. Any common factor, as m, may be expunged, except from the two extremes, or from the two means; for, if $\frac{m\,a}{b} = \frac{m\,a}{d}$, on dividing both sides by m, the resultant is $\frac{m\,a}{m\,b} = \frac{m\,c}{m\,d}$; *or*, $\frac{a}{b} = \frac{c}{d}$. Or, if $\frac{a}{m\,b} = \frac{c}{m\,d}$; on multiplying by m, we have $\frac{m\,a}{b\,m} = \frac{m\,c}{m\,d}$; or, again $\frac{a}{b} = \frac{c}{d}$. Or, if $\frac{m\,a}{m\,b} = \frac{c}{d}$; $\therefore \frac{m\,a}{m\,b} = \frac{c}{d}$; $\therefore \frac{m\,a}{m\,b} = \frac{a}{b}$; $\therefore \frac{m\,a}{m\,b} = \frac{c}{d}$ is identical with $\frac{a}{b} = \frac{c}{d}$.

RULE 6. In two or more series of Proportionals, the *products* of the corresponding antecedents and consequents also constitute a Proportion, and this is named a *Compound Proportion;* for if $\frac{a}{b} = \frac{c}{d}$ & $\frac{e}{f} = \frac{g}{h}$, on multiplying the terms, on the left hand of the sign $=$, together, and those on the right hand, $\frac{a}{b} \times \frac{e}{f}$ & $\frac{c}{d} \times \frac{g}{h}$, the resultant will be $\frac{a\,e}{b\,f} = \frac{c\,g}{d\,h}$.

Three or more series of proportionals are compounded in a similar way.

The *Essential Property*, or *Criterion*, of numbers in true proportion is, that *the product of the extremes is equal to the product of the means;* and reciprocally, if the product of any two numbers equals the products of any other two numbers, the series of numbers constitutes a proportion; thus,

in $15 : 5 :: 36 : 12$;— $15 \times 12 = 180 = 5 \times 36$; $\therefore$ a *true* proportion.
in $14 : 5 :: 35 : 12$;—$14 \times 12 = 168$; — $5 \times 35 = 175$; $\therefore$ a *false* proportion.

From the Essential Property of numbers in true proportion it follows, that *when any three terms are given the fourth may be found;* for on dividing the product of the means by the given extreme, or that of the extremes by the given mean, the other extreme, or the other mean, will be obtained; thus in $6 : 8 :: 24 : x$;

$\therefore 6\,x = 8 \times 24$; $\therefore x = \frac{8 \times 24}{6} = 32$, and the terms are $6 : 8 = 24 : 32$.

Or, in $7 : x :: 14 : 6$,

$\therefore 7 \times 6 = 14\,x$, $\therefore = \frac{7 \times 6}{14} = 3$; and the terms are $7 : 3 = 14 : 6$.

If each of the means is the same number, as in $3 : 6 :: 6 : 12$, the product of the extremes equals the square of one of the means; the value of *one of the*

means is therefore *the square root of the product* of the means; thus in $4 : x :: x : 16$,

$\therefore 4 \times 16 = x^2$, $\therefore x = \sqrt{64} = 8$; and the terms are $4 : 8 :: 8 : 16$.

When the value of one of the means equals the square root of the product of the extremes, that number is named a *mean proportional* between two given numbers. Stated algebraically, when $a : x = x : b$, then $x^2 = a\,b$; or $x = \sqrt{a\,b}$; x being the mean proportional between a and b.

There are *other important properties* of numbers in proportion resting on the equation which may be formed from four proportionals, $\frac{A}{B} = \frac{C}{D}$. For instance as shown above,—add 1 to both sides; $\frac{A}{B} + 1 = \frac{C}{D} + 1$, or $\frac{A+B}{B} = \frac{C+D}{D}$; which is the theorem known by the word *componendo*, putting together: *Or*, taking 1 from both sides; $\frac{A}{B} - 1 = \frac{C}{D} - 1$, *i.e.*, $\frac{A-B}{B} = \frac{C-D}{D}$; which is the theorem indicated by *dividendo*, taking apart. Again, on dividing *unity*, first, by $\frac{A}{B}$, and next, by $\frac{C}{D}$, we have $\frac{1}{1} \div \frac{A}{B} = \frac{B}{A}$, and $\frac{1}{1} \div \frac{C}{D} = \frac{D}{C}$; thus giving, *invertendo*, by inversion, $\frac{B}{A} = \frac{D}{C}$; And on multiplying each side by $\frac{B}{C}$, we have $\frac{A}{B} \times \frac{B}{C} = \frac{A}{C}$, and $\frac{C}{D} \times \frac{B}{C} = \frac{B}{D}$; i. e., *alternando*, by taking each other one, $\frac{A}{C} = \frac{B}{D}$.

These Properties, with their variations and combinations we now mention in their order.

1°. *Multiplicando*, or *dividendo*, by multiplying, or dividing by the same number, the two first terms or the two last,—the two antecedents or the two consequents,—the equation and consequently the proportion is undisturbed.

Take the Proportionals 9 : 12 :: 16 : 24;

	Multiplicando, by 3;	*Dividendo*, by 3.
1 & 2 terms,	27 : 36 :: 18 : 24;	3 : 4 :: 18 : 24;
3 & 4 „	9 : 12 :: 54 : 72;	9 : 12 :: 6 : 8;
1 & 3 „	27 : 12 :: 54 : 24;	3 : 12 :: 6 : 24;
2 & 4 „	9 : 36 :: 18 : 72;	9 : 4 :: 18 : 8;

In all these changes the product of the extremes remains equal to the product of the means, and the numbers resulting are therefore still proportionals. In fact the ratio of the two first terms $\frac{9}{12}$, or that of the two last $\frac{18}{24}$, is the same fractional number $\frac{3}{4}$,—and fractions undergo no change of value when the terms of the fraction are *both* multiplied or *both* divided by the same number. Also when the product of the extremes and that of the means are each multiplied or divided by the same number, the resulting products or quotients are also equal, and the proportion remains.

2°. *Alternando*, or *invertendo*, by taking the terms alternately, or by inverting the terms, the proportion also remains undisturbed;

Thus in the proportions $\frac{12}{4} = \frac{75}{25}$,

1st, exchange the extremes, $\frac{25}{4} = \frac{75}{12}$; the ratio being $\frac{25}{4}$,
2nd, „ means, $\frac{12}{75} = \frac{4}{25}$; „ „ $\frac{4}{25}$,
3rd, „ means & extremes, $\frac{4}{12} = \frac{25}{75}$; „ „ $\frac{1}{3}$.

But the proportions continue unbroken though the ratio in each case is different; for after the various exchanges the *criterion* is satisfied,—the product of the extremes *does* equal that of the means.

3°. *Componendo vel dividendo,* by putting the two first terms of the proportion together, and the two last; or by separating the two first and the two last,—there arises the proposition, that the *sum* or *the difference of the two first terms* is *to the second,* as *the sum* or *difference of the two other terms* is *to the fourth*: for example,

in the proportion $24 : 8 :: 45 : 15$, the *com. ratio* is 3;
componendo by adding $24 + 8 : 8 :: 45 + 15 : 15$, „ „ 4;
dividendo by subtracting $24 - 8 : 8 :: 45 - 15 : 15$, „ „ 2;

The explanation to be given is, that when we increase or diminish each antecedent by its consequent, we do nothing except increase or diminish by *unity* each of the two ratios; and since these ratios were equal at first, they remain equal after such an increase or diminution: for

changing the means, $24 \pm 8 : 45 \pm 15 :: 8 : 15$; (1)
but $24 : 8 :: 45 : 15$; (2)
or, $24 : 45 :: 8 : 15$; (3)

& ∵ two ratios equal to a third are equal to each other,

∴ $24 \pm 8 : 45 \pm 15 :: 24 : 45$;

or rather $24 \pm 8 : 24 :: 45 \pm 15 : 45$.

Thus *the sum,* or *the difference of the two first terms,* is *to the first term,* as *the sum,* or *difference of the two other terms,* is *to the third term.*

4°. Generally, *addendo vel subtrahendo,* by increasing, or diminishing *the sum,* or *the difference of the antecedents* is *to the sum,* or *the difference of the consequents,* as *any one of the antecedents* is *to its consequent.*

Take the proportion, $24 : 8 :: 45 : 15$;
exchanging the means, $24 : 45 :: 8 : 15$;
by the former property 3°, $24 \pm 45 : 45 :: 8 \pm 15 : 15$;
again exchanging the means, $24 \pm 45 : 8 \pm 15 :: 45 : 15$;
whence by reason of the common ratio $45 : 15$,
$24 + 45 : 8 + 15 :: 24 - 45 : 8 - 15$;
or, rather exchanging the means,
$24 + 45 : 24 - 45 :: 8 + 15 : 8 - 15$.

i. e., the sum of the antecedents is *to their difference,* as *the sum of the consequents* is *to their difference.*

Cor. Let there be a succession of numbers, forming, two by two, equal ratios, *the sum of all the antecedents* is *to the sum of all the consequents,* as *any one antecedent* is *to its consequent.*

Assume the series, $8 : 12 :: 2 : 3 :: 4 : 6$ &c., *or,* $a : b = c : d = e : f$, &c.

the ratios	$8 : 12 :: 2 : 3$,	$a : b :: c : d$;
by 4° give	$8 + 2 : 12 + 3 :: 2 : 3$;	$a + c : b + d = c : d$;
but	$2 : 3 :: 4 : 6$,	$c : d = e : f$;
∴	$8 + 2 : 12 + 3 :: 4 : 6$	$a + c : b + d :: e : f$;
applying 4°.	$8 + 2 + 4 : 12 + 3 + 6 :: 4 : 6$	$a + c + e : b + d + f = e : f$.

and so on, whatever may be the number of equal ratios.

If there be any number of fractions equal in value, as $\frac{8}{12}$, $\frac{4}{6}$, $\frac{2}{3}$, &c., if *the sum* or *the difference of the numerators* and *that of the denominators be taken the resulting fraction is equal in value to each of the given fractions*: thus,

in $\frac{8}{12}$, $\frac{4}{6}$, $\frac{2}{3}$,

$8 \pm 4 \pm 2 = 14$ or 2, and $12 \pm 6 \pm 3 = 21$ or 3,

the resulting fraction being $\frac{14}{21}$ or $\frac{2}{3}$.

The equation $\frac{8}{12} = \frac{2}{3}$ leads to the proportion $8 : 12 :: 2 : 3$;

whence by the foregoing property, $\dfrac{8+2}{12+3} = \frac{10}{15} = \frac{6}{9} = \frac{2}{3}$.

5°. COMPOUND RATIO; *Componendo*, by placing together or combining. In any number of proportions, *if all the corresponding antecedents and consequents be respectively multiplied together, the resulting products will be in proportion.* For example,

$$\begin{array}{lll} 3 : 8 :: 12 : 32, & & or,\ a : b :: c : d; \\ 7 : 15 :: 28 : 60, & & e : f :: g : h; \\ 40 : 12 :: 50 : 15, & & i : k :: l : m, \&c. \end{array}$$

being proportionals they may be represented by

$$\frac{3}{8} = \frac{12}{32}, \qquad or,\ \frac{a}{b} = \frac{c}{d},$$
$$\frac{7}{15} = \frac{28}{60}, \qquad \frac{e}{f} = \frac{g}{h};$$
$$\frac{40}{12} = \frac{50}{15}, \qquad \frac{i}{k} = \frac{l}{m}, \&c.;$$

on multiplying the corresponding sides of these equations, there will result the equal products,

$$\frac{3 \times 7 \times 40}{8 \times 15 \times 12} = \frac{12 \times 28 \times 50}{32 \times 60 \times 15} \qquad or,\ \frac{iae}{bfk} = \frac{cgl}{dhm};$$

$$i.e., \quad \frac{840}{1440} = \frac{16800}{28800}; \qquad \text{or, } 840 : 1440 :: 16800 : 28800,$$

satisfying the criterion of proportion that $840 \times 28800 = 1440 \times 16800$.

If we divide $\frac{16800}{28800}$ by 20, we have $\frac{840}{1440} = \frac{840}{1440}$, an *identical proportion*, the terms of the two ratios being the same.

N.B.—The *constant ratio* of the preceding proportion, namely $\frac{840}{1440}$ is equal to the product of the three constant ratios of the given proportions.

Thus the three constant ratios being $\frac{3}{8}$, $\frac{7}{15}$, $\frac{40}{12}$, or $\frac{10}{3}$, we have for their product $\frac{210}{360}$; or suppressing the common factor 30, $\frac{7}{12}$, to which the fraction $\frac{840}{1440}$ may be reduced by suppressing the common factor 120.

The ratio $\frac{7}{12}$, thus arising from the multiplication of several other ratios, is named the *compound ratio*.

6°. From the theory of compound proportion it follows, *when four numbers are in proportion*, 1st, *their squares, cubes, and other like powers are also in proportion*, and 2nd, *their square roots, cube roots, and other like roots are in proportion.*

1st. Take $a : b :: c : d$, or, $4 : 6 :: 8 : 12$;
squaring; $a^2 : b^2 :: c^2 : d^2$, $16 : 36 : 64 : 144$;
cubing, $a_3 : b_3 :: c^3 : d_3$, $64 : 216 : 512 : 1728$.

Thus there results a series of proportions which, multiplied in the order of the antecedents and consequents, will give products also in proportion.

2nd. Take $a^2 : b^2 :: c^2 : d^2$, or, $4 : 9 :: 16 : 36$;
extracting the $\surd$, $a : b :: c : d$, or, $2 : 3 :: 4 : 6$;

$\because$ the ratio $\frac{a^2}{b^2} = \frac{c^2}{d^2}$ or $\frac{4}{9} = \frac{16}{36}$, $\therefore$ the ratios of the roots are equal.

But to extract the square root of a fraction, as $\frac{4}{9}$ or $\frac{16}{36}$, we extract the square root of the numerator and that of the denominator, which gives $\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$ and $\frac{\sqrt{16}}{\sqrt{36}} = \frac{4}{6}$

7°. *Irrational numbers*, or numbers without a perfect root, and *Incommensurables.* When quantities as a, b, c, d, or numbers, as 2, 3, 8, 12, are *not* perfect squares, the quantities $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, $\sqrt{d}$, or $\sqrt{2}$, $\sqrt{3}$, $\sqrt{8}$, $\sqrt{12}$, are named *irrational* quantities or numbers; and when quantities or numbers are so related, that, although one of them may be represented in the terms of a certain unit, the other cannot be so represented, such quantities or numbers are named *incommensurable.* The diameter of the circle in relation to the circumference, and the diagonal of a square of which the side is unity in relation to that side, are familiar examples. Also in numbers, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{15}$, &c., are incommensurable with unity, for there is no mixed number, nor fraction, exactly equivalent to them. Proportion however exists between incommensurable numbers, for $\sqrt{2} : \sqrt{3} = \sqrt{8} : \sqrt{12}$; and we are led to consider *ratios*, which must in general be regarded as *irrational*, because between numbers without a perfect root: but by *ratio* we must bear in mind that we mean the *proportion* between quantities or numbers;—and by *irrational*, the non-existence of an exact root. Now the question arises, can we apply to proportions of this kind, *i.e.*, between irrational and incommensurable quantities or numbers, all the properties which have just been established ?

The answer is affirmative, if we remember that an irrational number may always be replaced *in the mind* by an exact fractional number which only differs from the proposed number by a quantity so small that, when we neglect it, we need not have any regard to the mistake committed; and it is between the commensurable numbers substituted for the irrational magnitudes that the ratios are judged to be established.

As to the ratios between exact fractional numbers, it is easy to understand, that according to the rule for the division of fractions. they may always be replaced by the ratios between whole numbers; thus, the ratio of $\frac{3}{7}$ to $\frac{5}{11}$, being the quotient arising from the division of $\frac{3}{7}$ by $\frac{5}{11}$ is equal to $\frac{3}{7} \times \frac{11}{5}$, or to $\frac{33}{35}$, that is to say, to the ratio of 33 to 35.

Again, the ratio of $\frac{7}{8}$ to $\frac{15}{23}$ is equal to $\frac{7}{8} \times \frac{23}{15}$, or to the ratio of 161 to 120.

Thus, all the properties of proportion are true with respect to numbers, whatever may be the numbers about which we reason. Similar truth and certainty will appear even in a higher degree when we consider;

EUCLID'S

THEORY OF GEOMETRICAL PROPORTION.

By some writers this fifth book of Euclid has been named *the Elements of Mathematical Logic*. The other books have shown us the absolute properties, the equality or inequality of Plane Geometrical figures;—*this* book enables us to institute a comparison between them. The Definitions and Propositions indeed are so expressed, as if they applied only to such plane figures; but they extend equally to lines, surfaces, amd solids, and to every species of quantity. Whatever be the science or branch of knowledge that depends upon Proportion, it is founded on the Principles contained in the fifth book of Euclid. We take, for instance, Practical Geometry;—nearly all its measurements are calculated by the Doctrines of Proportion,—the survey of an estate, or of a whole kingdom, is carried out by the application of those doctrines: and in Arithmetic, Astronomy, Statics, &c., the use of this book is indispensable. "In fine," says an old writer, "one may affirm, that if one should take away the knowledge of the Propositions that this Book giveth us, the remainder would be of little use."

The Definitions and Propositions of the fifth book may be extended to every species of quantity and magnitude; and may be easily applied to *number;* but though our clearest notions of ratio and proportion are derived from numbers, Number, as we have said before, is only to a small extent the actual subject of Geometry.

N.B.—The text of the Definitions and Propositions is from Simson's Edition;—variations are given in the notes, and if between brackets, or " " they are from the Greek text of Euclid.

DEFINITIONS.

I.—A less magnitude is said to be a *part* of a greater magnitude, when the less measures the greater; that is when the less is contained a certain number of times exactly in the greater.

"A part is a magnitude of a magnitude,—the less of the greater, when it exactly measures the greater."—EUCLID.

"For the clearer vnderstandyng of a parte, it is to be noted, that a part is taken in the Mathematicall Sciences two maner of wayes. One way a part is a lesse quantitie in respect of a greater, whether it measure the greater or no. The second way, a part is onely that lesse quantitie in respect of the greater which measureth the greater."—BILLINGSLEY, fol. 126.

By *part* is meant an *aliquot* part, or *submultiple,*—not any portion of a whole or greater magnitude, but *the* portion, which if repeated will exactly make up that whole, or greater magnitude; thus, if A represents a line $=$ 3 inches, and B a line $=$ 9 inches, A $=$ 3 is an aliquot part or submultiple of B $=$ 9;—A repeated exactly measures B, and B is the multiple of A. In numbers 4 and 6 are submultiples of 24; and the quantities, a, b, c, d, of $a\ b\ c\ d$.

Aliquant part has been used for the less magnitude, which though repeated does not exactly make up the greater; thus a measure of 3 feet is an *aliquant* part of 7 feet.

The magnitudes compared must be *of the same kind,*—lines, or surfaces, or solids,—weight, or time, &c. Lines bound surfaces, but are no part of them; an hour cannot be measured by an ounce,—and a penny is no submultiple of a mile.

One magnitude *measures* another when it is contained in that other magnitude an exact number of times.

And a magnitude which is a measure of two or more magnitudes is named the *common measure* of those magnitudes.

II.—A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, "when the greater contains the less a certain number of times exactly."

"The greater magnitude is a multiple of the less, when it is exactly measured by the less."—EUCLID.

"The *Multiplex* is a great quantity compared with a less which it contains precisely some number of times."—DE CHALES.

"By this worde *multiplex,*" says BILLINGSLEY, fol. 127, "which is a terme proper to arithmitike and number, it is easy to consider that there can be no exact knowledge of proportion and proportionalitie, and so of this fifth booke, wyth all the other bookes followyng, without the ayde and knowledge of numbers." If excuse be needed, this is the reason why we have prefixed, "*Some Properties of Proportional Numbers* to EUCLID's otherwise matchless "*Theory of Geometrical Proportion.*"

In things of the same kind every greater magnitude contains the less; thus a vessel of 27 cubic inches contains another vessel of 13 cubic inches;—but the greater is a multiple of the less only when it is measured exactly by the less,—*i. e.*, by its submultiple; thus a piece of cloth 27 yards in length is the multiple of another piece 9 yards in length,—3 repetitions of the less exactly making up the greater.

Equimultiples are magnitudes containing their respective aliquot parts the same number of times; thus *one yard* and *one barrel* are equimultiples of *an inch* and *a gallon;*—for the yard contains the inch 36 times, and the barrel also contains the gallon the same number of times.

All magnitudes for which an exact common measure, or submultiple, can be found are *Commensurable* magnitudes, as the radius and diameter of a circle, and the angle of an equilateral triangle, of a square, or of a reg. hexagon, and the space round any point: and those magnitudes are *Incommensurable* which have no common measure; thus the diagonal of a square represented in numbers, and the side of that square have no common measure;—for if the side contains 100 units, the diagonal will contain less than 142 and more than 141;—what is the submultiple of the one is not the submultiple of the other. The same is also true,—1°. of the diameter and circumference of a circle; 2°. of the diagonal and side of a cube; 3°. of the segments of a line cut in extreme and mean ratio. None of these have a common measure, neither have they a common multiple.

III.—*Ratio* is a mutual relation of two magnitudes of the same kind to one another in respect of quantity.

A mistake in translating Euclid's κατα πηλικότητα "in respect of quantity," has tended to confuse this definition. The *how great* one thing is when compared with another is the hinge on which the definition turns. Euclid is speaking of two magnitudes with respect to the spaces which they occupy, whether length, or area, or bulk, and his meaning therefore is better expressed by saying that "Ratio is the relative size which two magnitudes of the same kind have to one another with respect to the space which they occupy." A square of six acres in area, though greater than a square of four acres in area, is, when compared with four acres, a less magnitude than a square of three acres is, when compared with a square of one acre; the space which the six acres occupy is only one and a half times larger than the space which the four acres occupy,—but the space comprised in three acres is three times larger than the space comprised in one acre. Thus the *how great* one thing is, when compared with another, is the essential idea which belongs to EUCLID's definition of ratio.

Certainly *the how great* is best expressed by numbers; an algebraical notation, m times, or n times, or $\frac{m}{n}$ times, may denote generally that one magnitude B is m times A, or $\frac{m}{n}$ times A; but "the particular ratio of two given magnitudes, whether commensurable or otherwise, can be" expressed or "conceived only by means of the numbers which denote how often the same magnitude is contained, or nearly contained, in each." Without these numbers we form no idea of the relative magnitude of the two given magnitudes,—for numbers constitute, either its exact, or its proximate measure. *See* GEOM. PL. SOL. & SPHER. p. 32.

A ratio is expressed by *two terms*, as A : B, or 6 : 12; the foundation from which the comparison proceeds, A, is named the *antecedent*, and the term to which the comparison extends, B, the *consequent*.

In two commensurable magnitudes the *numerical ratio* of one to the other, "is a certain number, whole or fractional, which expresses *how many*, and *what* parts of the second are contained in the first: for example, if

the common measure of A and B be contained in A *five* times, and in B *six* times, or, which is the same thing, if A contains $\frac{5}{6}$ths of B, then A is said to have to B the numerical ratio '5 to 6,' which is written 5 : 6, or, in the fractional form, $\frac{5}{6}$."

The *measure of a ratio*, or how great one magnitude is when compared with another of the same kind, is determined, by ascertaining *how often* the first magnitude is contained in the second, or *what part* the first magnitude is of the second; if one line, A B, contains 12 units of length, and another line, C D, 4 of the same units, the measure of their relative magnitudes $= 12 \div 4$, or 3; and if one line, E F, contains 3 units, and another G H, 12 units, the measure $= 12 \div 3$, or $\frac{1}{4}$.

When $A = B$, the ratio A : B is one of *equality*; when A is $>$ B, the ratio is of *greater inequality;* and when A is $<$ B, it is of less *inequality.*

The *inverse* or *reciprocal ratio* arises from changing the order of its two terms; as, instead of A : B, or 5 : 6, making the ratio B : A; 6 : 5 or $\frac{6}{5}$.

IV.—Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

"Magnitudes are said to have a ratio to one another, which are able on being multiplied to exceed one another."—EUCLID.

"In Geometry, *multiplication* is only a repeated addition of the same magnitude, and *division* is only a repeated subtraction."—POTTS. The successive foldings up of a string, or of a piece of cloth, in folds of the same length, is geometrical multiplication; the successive unfoldings, or cuttings off, of pieces of the same length, is geometrical division.

By this definition is excluded the comparison between any two magnitudes of which one is *finite* and the other *infinite*;—for no addition of finite things can ever equal, much less exceed, the infinite; and for the same reason we cannot institute a comparison between two magnitudes of which one is infinitely small and the other infinitely large.

V.—*Definition of Proportion.*—The first of four magnitudes is said to have the *same ratio* to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

"Magnitudes are said to be in the *same ratio,* the first to the second, and the third to the fourth,—when the equimultiples of the first and third, being at the same time compared with the equimultiples of the second and fourth, each with each, are, whatever the multiplication may be, together greater, or together equal, or together less."—EUCLID.

The whole reasoning of the fifth book rests on this celebrated definition as its foundation;—it is indeed, *the* definition which supplies a *criterion* for determining the equality of two ratios.

We shall exemplify its meaning by taking four magnitudes A, B, C, D; and of A and C equimultiples E, G, five times A, C; and of B and D equimultiples F, H, twice B, D: also K, M, four times A, C; and L, N, double of B, D; finally, O, Q, thrice A, C; and P, R, four times B, D; now,

A.	B.	C.	D.
2.	4.	3.	6.
E.	F.	G.	H.
10.	8.	15.	12.
K.	L.	M.	N.
8.	8.	12.	12.
O.	P.	Q.	R.
6.	16.	9.	24.

$\because$ E $>$ F, and G $>$ H; K $=$ L, and M $=$ N; O $<$ P, and Q $<$ R;
$\therefore$ A : B $=$ C : D.

Or, Let $a : b = c : d$; and let a, c, each be taken m times, b and d each n times;

then, 1. If $m\,a > n\,b$; also $m\,c > n\,d$;
2. If $m\,a = n\,b$; „ $m\,c = n\,d$;
3. If $m\,a < n\,b$; „ $m\,a < n\,d$.

Thus the *test* of the equality of the ratios $a : b$ and $c : d$, is, that any equimultiples of a and c are always greater than, equal to, or less than the equimultiples of b and d.

The excellence of Euclid's definition consists in its *applicability* to *all* magnitudes, to *incommensurables* as well as to *commensurables*. See *Study of Mathematics*, pp. 85, 86. Applied to incommensurables indeed the second part of the test does not occur,—for in that case no multiple of one quantity can be found exactly equal to a multiple of the other; but when quantities are commensurable, the second part is as applicable as the first and third.

Lardner's comment, in elucidating the meaning of the text, establishes that *Ratios are the same*, 1°. if their consequents are equimultiples of their antecedents; 2°. if their antecedents be equimultiples of their consequents; 3°. if any equimultiples whatever of the antecedents are also equimultiples of their consequents; or 4° when the ratios of every pair of equimultiples of their antecedents to every pair of equimultiples of their consequents are ratios of the same species of inequality,—*i. e.*, if A and a be multiplied by the same number, the results must be either both greater, equal to, or less than, the results obtained by multiplying B and b by any number.

"The best commentary," says the *Manual of Euclid, pt.* II., *p*. 29, "is in the first and last Propositions of the Sixth Book;" where in applying the test of equality of ratios, it is shewn from the nature of the case considered, that what is proved of one set of multiples must necessarily be true of all.

VI.—Magnitudes which have the same ratio are called *proportionals.*

The same ratio is expressed by the sign :: or $=$; thus A : B :: C : D, or A : B $=$ C : D; or in the manner of a fraction $\frac{A}{B} = \frac{C}{D}$, indicating the division of the antecedent by the consequent.

VII.—When of the equimultiples of four magnitudes, (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a *greater ratio* than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a *less ratio* than the first has to the second.

A *greater ratio* exists, if the first magnitude contains any aliquot part of the second a greater number of times than the third contains the like aliquot part of the fourth; for example, in the four quantities, a, b, c, d, let $ma > n\,b$, but $m\,c < n\,d$, then $a : b > c : d$; or, let $m\,a < n\,b$ but $m\,c > n\,d$ then $a : b < c : d$. Or, take four numbers 101, 10, 200, 20; then, $\because$ 101 contains one hundred and one times the tenth part of 10,— but 200 contains only one hundred times the tenth part of 20, $\therefore$ the ratio $101 : 10 >$ the ratio $200 : 20$; and $\therefore$ 101 is a greater number with respect to 10 than 200 is with respect to 20.

VIII.—Analogy or Proportion is the similitude of the ratios.

"Analogy is the sameness (identity) of the ratios."—EUCLID.

Analogy is a *reasoning out* of two sets of comparisons and a declaring of their identity; thus, the admiral and the fleet are one set of relations,— the general and the army another,—and our reasoning is an analogy, when we declare, *as* an admiral to his fleet, *so* is a general to his army.

Ratio is the comparison instituted between two magnitudes,—Analogy, the comparison between two ratios,—the analogy being complete or perfect when the ratios are identical. Take four magnitudes A, B, C, D, and from them form the ratios A : B and C : D; if the one ratio equals the other, there exists a proportion between the magnitudes,—A : B :: C : D. This proportion does not establish the absolute, but the relative magnitudes.

IX.—Proportion consists in three terms at least.

The middle term being repeated, there are in reality four terms, without which a proportion cannot be established; thus, $a : b :: b : c$, or $2 : 4 :: 4 : 8$.

X.—When three magnitudes are proportionals, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

A duplicate, or two-fold ratio is a ratio compounded of two equal ratios; thus if $a : b : c$, then $a : c = \frac{a}{b} \times \frac{b}{c}$; or $2 : 4 : 8$; then $\frac{2}{8} = \frac{2}{4} \times \frac{4}{8} = \frac{8}{32} = \frac{1}{4}$.

The duplicate ratio is expressed *algebraically* and *arithmetically* by the ratio of their squares; thus, $a : b : c$;— $a : c = a^2 : b^2$,
or $2 : 4 : 8$; $2 : 8 = 4 : 16$.

Such magnitudes are in *continued proportion*;—for magnitudes are continued proportionals, when every two terms have always the same ratio; or when the first has the same ratio to the second as the second has to the third,—and the second to the third the same as the third to the fourth; thus, $a : b : c : d : e$, &c., or $2 : 4 : 8 : 16 : 32$, &c., the common ratio being 2.

In continued proportionals the first and last terms are the *extremes*, the intermediate are the *means*.

A *mean proportional* is a magnitude between two other magnitudes, forming with them a continued proportion; and the *third proportional* is the magnitude in continued proportion with the two other magnitudes.

A *double Ratio* and a *duplicate Ratio* must not be confounded. A double, triple, &c., Ratio is so called when the antecedent is double, triple, &c., of the consequent; a duplicate Ratio is a ratio compounded of two equal ratios, as in the proportionals, $2 : 4 : 8$, or $3 : 9 : 27$;—$2 : 8$, or $3 : 27$ being compounded of $2 : 4$, and $4 : 8$;—or of $3 : 9$, and $9 : 27$;—in the one instance it is the half of a half,—in the other the third of a third. The half of a half is the square of $\frac{1}{2}$,—the third of a third is the square of $\frac{1}{3}$; 2 is $\frac{1}{2}$ of the $\frac{1}{2}$, or $\frac{1}{4}$ of 8,—and 3 is the $\frac{1}{3}$ of the $\frac{1}{3}$, or $\frac{1}{9}$ of 27. In like manner $8 : 2$ is a duplicate ratio of $8 : 4$ because 8 is the double of 4.

Duplicate ratio is a species of compound ratio, of which instances occur in propositions 19 and 20, bk. vi.

XI.—When four magnitudes are continual proportionals, the first is said to have to the fourth, the *triplicate ratio* of that which it has to the second, and so on, *quadruplicate*, &c., increasing the denomination still by unity in any number of proportionals.

"When four magnitudes are proportional, the first is said to have to the fourth a ratio triplicate of that which it has to the second; and so on successively in order, as far as the analogy (or proportion) may extend." —EUCLID.

Triplicate ratio, compounded of three equal ratios, is the ratio of the cubes; thus, if $A : B : C : D$, or $2 : 4 : 8 : 16$;—then $A : D = A^3 : B^3$, or $2 : 16 = 8 : 64$. The ratio $2 : 16$ is triplicate of the ratio of $2 : 4$; for 2 is $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$, or $\frac{1}{8}$ of 16.

Triplicate ratio is a species of compound ratio, instances of which are found in the 11th and 12th books of Euclid on the Geometry of Solids.

A numerous list of the kinds of proportion might be given; as, Duple proportion, triple, quadruple, quintuple,—Sesquialter, sesquitertia, sesquiquarta, — Super-partiens, super-bipartiens, super-tripartiens, &c., — Multiplex, super-particular, &c., &c.;—but most of them are rather ingenious puzzles than of practical utility.

DEF. A—*Of Compound Ratio.*—When there is any number of magnitudes of the same kind, the first is said to have to the last of these the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D, be four magnitudes of the same kind, the first, A, is said to have to the last, D, the ratio compounded of the ratio of A to B, and of the ratio B to C, and of C to D; or the ratio of A : D is said to be compounded of the ratios of A : B, B : C, and C : D.

So 4, 5, 3, 11 are four numbers, and $4 : 11 = \frac{4}{5} \times \frac{5}{3} \times \frac{3}{11} = 60 : 165$.

And if A : B = E : F, B : C = G : H, C : D = K : L; then A : D is a ratio compounded of ratios, which are the same with the ratios of E : F, G : H, and K : L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L; thus,

If A : B = E : F; B : C = G : H; & C : D = K : L; then, A : D = E : F, of G : H, of K : L.

Or, if $3 : 4 = 6 : 8$; $4 : 5 = 8 : 10$; and $5 : 7 = 10 : 14$;
then, $\frac{3}{7} = \frac{6}{8}$ of $\frac{8}{10}$ of $\frac{10}{14} = 6 : 14$.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L. Thus

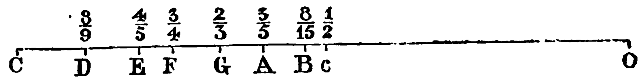
If A : D = M : N; then M : N = E : F, of G : H, of K : L.

Or, if $3 : 7 = 21 : 49$; then $21 : 49 = \frac{6}{8} \times \frac{8}{10} \times \frac{10}{14} = \frac{480}{1120} = 3 : 7$.

It may be observed, 1°., that the product of the fractions which represent numerical ratios corresponds to the compound ratios of magnitudes; 2°. in ratios compounded by this definition, the second term of each ratio is the same as the first term of the following ratio,—the consequent of the former becoming the antecedent of the latter.

DEF. B—There is a species of *Progression in the lengths of chords* of the same thickness and degree of tension, which produce the musical sounds of a certain note, of its fifth, and of its octave. Thus, if a musical string, C O, be divided so that its parts are in the proportion to one another as the numbers, $1, \frac{8}{9}, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{8}{15}, \frac{1}{2}$, the

vibrations of the respective parts, C O, D O, E O, F O, G O, A O, B O, *c* O, will yield the eight sounds



to which musicians give the names, C, D, E, F, G, A, B, *c*. Of these parts of a musical string, thus divided, it is the property, that *the first* is to *the third*, as *the difference between the first and second* is *to the difference between the second and third.* Thus $1 : \frac{4}{5} = \frac{1}{9} : \frac{4}{45}$.

Three straight lines, therefore, are said to be in *Harmonical Progression*, when the first is to the third as the difference of the first and second is to the difference of the second and third.

And when three lines, C O, D O, and E O are in harmonical Progression, D O is named a *harmonical mean* between C O and E O; and E O is named the *third harmonical progressional* to C O and D O.

In the same way magnitudes of any other kind are in harmonical progression, when the first : third : : the dif. between the first and second : the difference between the second and third; thus,

$$A : B = A \sim B : B \sim C.$$

And *any number* of lines or magnitudes are in harmonical progression, when every successive three are in harmonical progression.

XII.—In proportionals the antecedent terms are called *homologous* to one another,—as also the consequents to one another.

"Magnitudes are said to be homologous,—the antecedents to the antecedents, and the consequents to the consequents."—EUCLID.

Homologous magnitudes are those which correspond in the proportion; thus in A : B = C : D, A and C being antecedents, are homologous; and B and D, being consequents; but in A : B : C : D, A and C, B and C, B and D are homologous,—for the full statement of the progressionals is, A : B=B : C = C : D,—where B and C are at one time consequents, and at another time antecedents.

TECHNICAL WORDS TO DENOTE CHANGES IN THE ORDER OF PROPORTIONALS.

1. FOR FOUR PROPORTIONALS.

XIII.—*Permutando*, or *alternando*, by permutation, or alternately. This word is used when there are four proportionals, and it

is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth; as is shown in Prop. 16 of this fifth book. Thus,

If $A : B = C : D$, or $1 : 2 = 3 : 6$;
then *alternando* $A : C = B : D$; or $1 : 3 = 2 : 6$.

"Alternate ratio is the comparison of the antecedent to the antecedent, and of the consequent to the consequent."—EUCLID.

XIV.—*Invertendo*, by inversion; when there are four proportionals, and it is inferred that the second is to the first, as the fourth to the third. Prop. B. Book V. Thus,

If $\frac{A}{B} = \frac{C}{D}$, or $\frac{1}{2} = \frac{3}{6}$; then *invertendo*, $\frac{B}{A} = \frac{D}{C}$, or $\frac{2}{1} = \frac{6}{3}$

"Inverse ratio is the taking or comparing of the consequent as antecedent with the antecedent as consequent."—EUCLID.

In all changes of the order of Proportionals it is *essential*, that if one of the means be exchanged for an extreme, the other mean must also change places with the other extreme; and that if one of the extremes be placed as a mean, the other extreme must be placed as the other mean; thus, if $A : B = C : D$, or $1 : 2 = 3 : 6$; then $A : C \neq D : B$, or $\frac{1}{3} \neq \frac{6}{2}$.

XV.—*Componendo*, by composition; when there are four proportionals, and it is inferred that the first together with the second, is to the second as the third together with the fourth, is to the fourth. Prop. 18. Book V. Thus,

If $A : B = C : D$; then $A + B : B = C + D : D$.
Or, $1 : 2 = 3 : 6$; then $1 + 2 : 2 = 3 + 6 : 6$.

"The Synthesis, or Composition of ratio, is the taking of the antecedent along with the consequent as one term, in comparison with the consequent."—EUCLID.

Generally, the sum of the first *and* second is to the first *or* second, as the sum of the third *and* fourth to the third *or* fourth; thus, if $\frac{A}{B} = \frac{C}{D}$ then $\frac{A + B}{A \text{ or } B} = \frac{C + D}{C \text{ or } D}$; or, if $\frac{2}{3} = \frac{4}{6}$, then $\frac{5}{2} = \frac{10}{4}$, and $\frac{5}{3} = \frac{10}{6}$.

XVI.—*Dividendo*, by division; when there are four proportionals, and it is inferred that the *excess* of the first above the

second is to the second, as the *excess* of the third above the fourth is to the fourth. Prop. 17, Bk. V. Thus,

If $A : B = C : D$, A being greater than B, and C than D; then $A - B : B = C - D : D$;

or, if $3 : 1 = 6 : 2$; then, $3 - 1 : 1 = 6 - 2 : 2$.

"The *Diæresis* or Division of ratio is the taking of the excess whereby the antecedent exceeds the consequent in comparison with the consequent." EUCLID.

Geometrical Division, as we have seen, is the successive subtraction of a less magnitude from a greater.

Generally also,—the difference of the first *and* second is to the first *or* second, as the difference of the third *and* fourth to the third *or* fourth; thus, if $\frac{A}{B} = \frac{C}{D}$, or $\frac{2}{6} = \frac{8}{24}$; then, $\frac{A \sim B}{A \textit{ or } B} = \frac{C \sim D}{C \textit{ or } D}$;

$$Or,\ \frac{2 \sim 6}{2 \textit{ or } 6} = \frac{8 \sim 24}{8 \textit{ or } 24}.$$

XVII.—*Convertendo* by conversion; when there are four proportionals, and it is inferred that the first is to the *excess* above the second, as the third to its *excess* above the fourth. Prop. E., Bk. V. Thus, if $A : B = C : D$, or $2 : 1 = 6 : 3$; then $A : A - B = C : C - D$, or $2 : 2 - 1 = 6 : 6 - 3$.

"An *anastrophe* or *reversion* of the ratio is the taking of the antecedent in comparison with the excess whereby the antecedent exceeds the consequent."—EUCLID.

Combining several definitions into one,—*conjungendo*, the sum or difference of the first and second is to the first or second, as the sum or difference of the third and fourth is to the third or fourth;—*or* the sum of the first and second is to their difference, as the sum of the third and fourth to their difference.

2°.—FOR ANY NUMBER OF PROPORTIONALS ABOVE TWO.

XVIII.—*Ex æquali* (*sc. distantia*) or *ex æquo*, from equality of distance; when there is any number of magnitudes more than two, and as many others such that they are proportionals when taken two and two of each rank; and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others. Of this there are the two kinds, in definitions 19 and 20, which arise from the different order in which the magnitudes are taken two and two.

"The *ratio of equal distance*, (or of intervals) is when there are several magnitudes and others equal to them in number, taken two and two in the same ratio, and it arises that, as in the first set of magnitudes the first is to the last, so in the second set of magnitudes the first is to the last." "Otherwise, a taking of the extremes for comparison by a removal of the means."—EUCLID.

Thus, assuming $a : b = d : e$; $b : c = e : f$; $r : s = k : y$; $s : t = y : z$;
1st Series. $a, b, c \ldots r, s, t$, or 2, 4, 8, ... 6, 12, 24;
2nd ,, $d, e, f, \ldots k, y, z$, 3, 6, 12, ... 9, 18, 36;
then $a : t = d : z$, or 2 : 24 = 3 : 36.

XIX.—*Ex æquali*, or *ex æquo ordinate*. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank: so is the second to the third of the other; and so on in order; and the inference is as mentioned in the preceding definition; hence this is called *ordinate proportion*. It is demonstrated in Prop. 22, Bk. V.

"*Arranged Analogy*, or *Proportion* is when antecedent is to consequent, as antecedent to consequent; and also as consequent to some one thing, so is consequent to some other thing."—EUCLID.

Let A : B : C : D, 12 : 6 : 18 : 36; and E : F : G : H, 6 : 3 : 9 : 18; then A : D :: E : H; 12 : 36 = 6 : 18.

XX.—*Ex æquali in proportione perturbatâ seu inordinatâ*, or *ex æquo perturbate*, from equality in perturbate or disorderly proportion. Archimedes *de sphærâ et cylindro*. Prop. IV, Lib. 2. This term is used when the first magnitude is to second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank so is the third from the last to the last but two of the second rank; and so on in a cross order; and the inference is as in the 18th definition. It is demonstrated in Prop. 23, Bk. V.

"*Disturbed Analogy*, or Proportion is when, on there being three magnitudes and others equal to them in number, it comes to pass, that, as in the first magnitudes, antecedent is to consequent, as in the second magnitudes antecedent to consequent; so, in the first magnitudes, as the consequent is to some other thing, so, in the second magnitudes some other thing is to the antecedent."—EUCLID.

Let one series of magnitudes be A, B, C, 4, 8, 2; and another D, E, F, 12, 3, 6;

so that	A : B = E : F,	or	4 : 8 = 3 : 6 ;
and	B : C = D : E,	and	8 : 2 = 12 : 3 ;
then	A : C = D : F ;	then	4 : 2 = 12 : 6 ;

i.e,, the magnitudes being taken in a cross order are therefore said to be in disturbed or disordered Proportion, though in reality the Proportion is as exact as in any other case of proportion.

" Both this and the former inference come under one general principle, *scil.*, that ratios which are compounded of equal ratios are equal."— LARDNER.

The definitions, *ex æquo ordinate* and *ex æquo perturbate*, may readily be extended to any number of magnitudes, compared with an equal number of other magnitudes.

POSTULATES.

Let it be granted,

1.—That a given magnitude may be so increased that any required multiple of it may be taken.

2.—That any given multiple of a magnitude may be divided into parts, each of which is equal to that magnitude.

AXIOMS.

1.—Equimultiples of the same, or of equal magnitudes, are equal to one another.

2.—Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

3.—A multiple of a greater magnitude is greater than the same multiple of a less.

4.—That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

Additional Algebraic Expressions, &c.

N.B.—The capital letters A B, CD, E F, &c; or A, B, C, D &c., denote either lines, or other magnitudes of a like kind.

An *M* denotes magnitude,—*Ms*, magnitudes.

m	multiple.	n	another multiple.
m A &c.	multiple of A, &c.	$m+n$	the sum of the quantities m & n.
m A, m B, &c.,	equimultiples of A,B, &c.	$m\,n$	the product of $m \times n$.
$m\,(A+B)$	multiple of $(A+B)$.	$m\,n\,A$	a multiple of A by $m\,n$.
$m\,(A-B)$	multiple of $(A-B)$.	$(m+n)\,A$	„ „ A by $(m+n)$.
$m\,(A+B-C)$	multiple of the excess of $(A+B)$ above C.	*pt.*	part.
		sub-m.	submultiple.

The signs $>$, $\not>$, $<$, $\not<$, between ratios, as $A:B > C:D$, or $A:B \not> C:D$, or $A:B \not< C:D$, or $A:B < C:D$, denote that he one ratio is less than, or not less than, greater or not greater than the other, according to the sign.

Prop. I.—Theor.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same shall all the first magnitudes be of all the other.

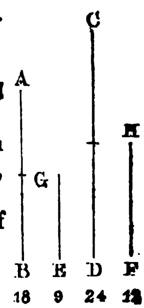
Con. Pst. 2, V.—Any given multiple of a magnitude may be divided into parts each of which is equal to that magnitude.

Dem. Ax. 2, V.—Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

Def. 2, V.—A greater magnitude is said to be a multiple of a less, when the greater is measured by the less.

Case I.—*Let the number of magnitudes in each set be two.*

E.1	Hyp.	Let A B, C D, &c., be equims. of, E, F, &c., each of each,
2	Conc.	then mult. A B of E = mult. (A B + C D) of (E + F).
D.1	Hyp.	∵ A B, C D equims. of E, F;
2	Ax. 2.	∴ mags in A B, each = E, *equal* Ms, in C D, each = F.
C.	Pst. 2, V.	Divide A B into A G, G B, each = E, and C D into C H, H D, each = F.
D.3	H.	∴ no. of Ms C H, H D = no. of other Ms A G, G B.
4	C.&Ax. 2, V.	∵ A G = E, and C H = F, ∴ A G + C H = E + F;
5	C.&Ax. 2, V.	and ∵ G B = E, and H D = F, ∴ G B + H D = E + F;
6	Conc.	∴ no. of Ms in A B, each = E *equal* no. of Ms in A B + C D, each = E + F.
7	Def. 2, V.	∴ the mult. A B is of E, that same mult A B + C D is of E + E.



Case II.—*Let the number of magnitudes in each set be more than two*; the same demonstration, which has been applied to two, holds for any number of magnitudes.

Conc. | $\because$ *If any number of magnitudes.* &c. Q. E. D.

Alg. & Arith. Hyp.—Let A = 24, B = 21, C = 18, &c., be equimults. say m times, or 3, of $a = 8$, $b = 7$, & $c = 6$.

Alg.—Then $\because$ $A = ma = a + a + a$; $B = mb = b + b + b$;
& $C = mc = c + c + c$;
+ the Equals, $A + B + C = ma + mb + mc = m(a + b + c)$;
$\therefore$ A + B + C same mult, m, of $a + b + c$, as A, B, C, are respectively of a, b, c.

Arith.—$\because$ $24 = 3 \times 8 = 8 + 8 + 8$; $21 = 3 \times 7 = 7 + 7 + 7$;
& $18 = 3 \times 6 = 6 + 6 + 6$;
+ the Equals; $24 + 21 + 18 = (3 \times 8) + (3 \times 7) + (3 \times 6)$
$= 3(8 + 7 + 6)$;
$\therefore$ 24 + 21 + 18, or 63, the same mult. of 8 + 7 + 6, or 21, that 24 is of 8, 21 of 7, and 18 of 6.

Cor.—Hence, if m be any number, $m\text{A} + m\text{B} + m\text{C} = m(\text{A} + \text{B} + \text{C})$, i.e., *the sum of the equimultiples = the equimultiple of the sum.*

Sch.—*If to a multiple of a magnitude by any number a multiple of the same magnitude by any number be added, the sum will be the same multiple of that magnitude that the sum of the two numbers is of unity.*

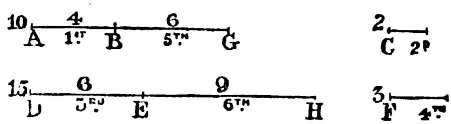
E. 1	Hyp.	Let $A = m\,C$ and $B = n\,C$;
2	Conc.	then $A + B = (m + n)\,C$.
D. 1	H. 1, V.	$\because$ $A = m\,C$, $\therefore$ $A = C + C + C$ &c. repeated m times;
2	H. 1, V.	and $\because$ $B = n\,C$, $\therefore$ $B = C + C + C$ &c. „ n „ ;
3	*Add*, Ax. 2, I.	Adding equals, $A + B = C$ taken $m + n$ times; i.e. $A + B = (m + n)\,C$;
4	Conc.	$\therefore$ A + B contains C as often as there are units in $m + n$.

Cor. 1—Thus, if there be any number of multiples whatever, as $A = m\,E$, $B = n\,E$, $C = p\,E$ &c., it is shown that $A + B + C = (m + n + p)\,E$.

Cor. 2.—Hence also, $\because$ $A + B + C = (m + n + p)\,E$;
and $\because$ $A = m\,E$, $B = n\,E$, and $C = p\,E$;
$\therefore$ $m\,E + n\,E + p\,E = (m + n + p)\,E$.

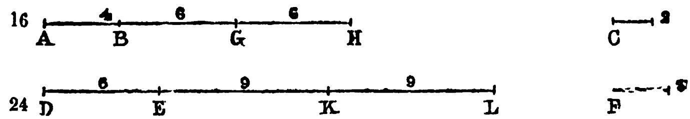
Prop. II.—Theor.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first, together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.



E.1	Hyp. 1	Let A B the 1st $= m$ C the 2nd, & D E the 3rd $= m$ F the 4th;
2	„ 2	& B G the 5th $= n$ C the 2nd, and E H the 6th $= n$ F the 4th;
3	Conc.	then A G = A B + B C $= (m + n)$ C; & D H = D E + E H $= (m + n)$ F.
D.1	Hyp. 1	∵ AB $= m$ C, and DE $= m$ F;
2	Def. 2, V. obs.	∴ magns. in AB (each = C) = magns. in DE (each = F);
3	*Sim*	So magns. in BG (each = C) = magns. in EH (each = F);
4	Ax. 2	∴ magns. in A G (each = C) = magns. in DH (each = F);
5	Def. 2, V.	∴ A G same mult. of C that D H is of F;
6	Remk.	i. e. A G (the 1st + 5th) same mult. of C the 2nd, that D H (the 3rd + 6th) is of F the 4th.
7	Rec.	*If therefore the first be the same, &c.,* Q. E. D.

Cor.—Hence, if any no. of magns. AB, BG, GH, be mults. of C, and as many, DE, EK, KL, the same *ms* of F, each of each;
then, A H i.e. (AB + BG + GH) the same *m* of C that DL, i. e. (DE + EK + KL) is of F.



Alg. & Arith. Hyp.—Take 6 quantities A = 24, B = 8, C = 21, D = 7, E = 16, F = 14, $m = 3$, and $n = 2$.

Alg. ∵ A the 1st = m B, and C the 3rd = m D.
E the 5th = n B, and F the 6th = n D;
+ *equals*, then A + E = $(m + n)$ B and C + F = $(m + n)$ D.
Now A + E contains B, $(m + n)$ times, & C + F contains D, $(m + n)$ times;
∴ Def. 2, V. A + E is the same mult. of B that C + F is of D.

Arith. ∵ $24 = 3 \times 8$, and $21 = 3 \times 7$;
$16 = 2 \times 8$, „ $14 = 2 \times 7$:
+ *equals*, $24 + 16 = (3 + 2)\,8$, and $21 + 14 = (3 + 2)\,7$.
But 40 contains 8, *five* times, and 35 contains 7, *five* times.
∴ Def. 2, V. 40 the same mult. of 8 that 35 is of 7.

Schol.—Allied to this Proposition is the Theorem" : *If the first of three magnitudes contain the second as often as there are units in a certain number;—and if the second contain the third also as often as there are units in a certain number, the first will contain the third as often as there are units in the product of these two numbers.*"

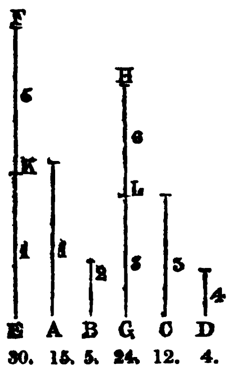
E. 1	Hyp.	Let A = m B, and B = n C;
2	Conc.	then A = mn C.
D. 1	H.	∵ B = n C, ∴ m B = $m\,(n\,C + n\,C$ &c.);
2	2, V.	but $m\,(n\,C + n\,C \text{ \&c}) = m\left\{(n + n + \text{\&c.})\,C\right\}$,
3	Remk.	and $n \times m = mn$;
4	Ax. 1, V.	∴ m B = mn C.
5	H. Ax. 1, I.	But A = m B, ∴ A = mn C.

Prop. III.—Theor.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

CON. Pst. 2, V.—DEM. Def. 2, V.—Cor. 2, V.

E. 1	Hyp. 1.	Let A the 1st $= m$ B the 2nd; and C the 3rd $=$ m D the 4th;
2	„ 2.	Also, let E F $= n$ A, and G H $= n$ C;
3	Conc.	then E F the same m of B as G H is of D.
D. 1	Hyp. 2.	∵ E F same m of A as G H of C;
2	Def. 2, V.	∴ as many magns. in E F, each $=$ A, as in G H, each $=$ C.
C. 1	Pst. 2, V.	Divide E F into E K, K F, each $=$ A,
2	„ V.	and GH into GL, LH, each, $=$ C;
3	Conc.	the no. of *Ms* in E K, K F $=$ no. of *Ms* in G L, L H.
D. 3	Hyp. 1.	∵ A same m of B, that C is of D;
4	C. 1.	and ∵ EK $=$ A, and GL $=$ C;
5	Def. 2, V.	∴ EK same m of B, that GL is of D;
6	*Sim.* 1.	and ∴ KF same m of B, that LH is of D.
7	„ 2.	and so, if more *Ms* in EF & GH each $=$ A, C.
8	Cor, 2, V.	Hence ∵ the 1st EK same m of the 2nd B, as 3rd GL, of 4th D;
9		and ∵ the 5th KF same m of the 2nd B, as 6th LH of 4th D;
10	2, V.	∴ EF (1st + 5th) same m of the 2nd B, as GH (3rd + 6th) of 4th D.
11	Conc.	If, *therefore the first be the same, &c.*



Q. E. D.

COR. If A, A′ be equimults. of B, B′ and also of C, C′; and if B be a m of C, the other B′ shall be the same m of C′.

Alg. & Arith. Hyp.—Take $a = 12$, $b = 4$, $c = 15$, $d = 5$; $m = 3$ & $n = 2$.

Alg.—Let $a = mb$, and $c = md$; then $na = mnb$, and $nc = mnd$; *i. e.*, the equims. $n\,a$ & $n\,c$ of the 1st and 3rd, are mults. of the 2nd and 4th.

Arith. $\because 12 = 3 \times 4$, and $15 = 3 \times 5$, $\therefore 2 \times 12 = 6 \times 4$, & $2 \times 15 = 6 \times 5$;
i. e. the equims. 24 & 30, of the 1st 12, and 3rd 15, are equims. of the 2nd 4, and of the 4th 5.

SCH.—"If any equimultiples m A, m C, be taken of the antecedents of an analogy, A : B :: C : D, and any equimultiples, n B, n D, of the consequents, these multiples, taken in the order of the terms, are proportional," i. e. m A : n B :: m C : n D.

C. 1	Pst. 1, V.	Of m A, m C take equims. p times, and of n B, n D, equims. q times;
D. 1	3, V.	then $\because$ m A, m C, contain A and C, p m units of times;
2	Conc.	$\therefore$ equims. m A, m C by p are equims. of A and C, and equal $p\,m$ A, $p\,m$ C.
3	*Sim*	So n B $\times$ q and n D $\times$ q = $q\,n$ B and $q\,n$ D.
4	H. and D. 2.	Since A : B :: C : D, and equims. of A and C are $p\,m$ A, $p\,m$ C,
5	D. 3.	and $\because$ equims. of B, D are $q\,n$ B, $q\,n$ D,
6	Def. 5, V.	$\therefore$ if $p\,m$ A $>$ $q\,n$ B, $p\,m$ C $>$ $q\,n$ D; if =, =, and if <, <.
7	D. 2.	But $p\,m$ A, $p\,m$ C are also equims. of m A and m C,
8	D. 5.	and $q\,n$ B, $q\,n$ D also equims. of n B and n D;
9	Def. 5, V.	$\therefore$ m A : n B :: m C : n D;

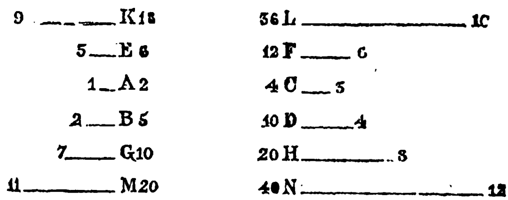
COR. When $n = 1$, then m A : B = m C : D.

PROP IV.—THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equal multiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; viz., "the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

CON.—Pst. 1, V. DEM. 3, V. Def. 5, V.

E. 1	Hyp. 1.	Let A : B = C : D;
2	„ 2.	and E, F be any equims. of A and C,
3	„ 3.	and G, H any equims. of B and D;
4	Conc.	then E : G = F : H.



C. 1	Pst. 1, V.	Of E and F take any equims. K, L,
2	,,	and of G and H ,, ,, M, N.
D. 1	Hyp. 2	∵ E is the same m of A as F of C,
2	C. 1.	and K, L are equims. of E and F;
3	3, V.	∴ K same m of A, that L is of C;
5	*Sim.*	So M same m of B, that N is of D.
5	Hyp. 1.	And ∵ $A : B = C : D$;
6	D. 3.	and K, L are equims. of A and C;
7	D. 4.	and M, N equims. of B and D;
8	Def. 5, V.	∴ K, $>, =$ or $<$ M, so L $>, =$ or $<$ N.
9	C. 1.	But K, L are equims of E and F;
10	C. 2.	and M, N equims. of G and H;
11	Def. 5, V.	∴ $E : G = F : H$;
12	Conc.	Therefore *if the first of four magnitudes &c.*

Q. E. D.

Cor. 1.—Likewise, *if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth;* and in like manner, *the first and third shall have the same ratio to any equimultiples whatever of the second and fourth.*"

Or, "If 4 Ms be proportional, then I°, any equims. being taken of the 1st and 3rd, the m of the 1st : 2nd = m of 3rd : 4th; and II°, any equims. being taken of 2nd and 4th, the 1st : m of 2nd = 3rd : m of the 4th."

E. 1	Hyp. 1.	Let $A : B = C : D$
2	Hyp. 2.	and let E, F be any equims. of A & C;
3	Conc.	then $E : B = F : D$.
C.	Pst. 1, V.	take of E, F any equims. K, L, and of B, D equims. G, H.

D. 1	4, V.	As before K same m of A, that L is of C.
2	Hyp. 1.	And $\therefore$ A : B = C : D,
3	C.	and K, L are equims, of A & C,
4	C.	and G, H are equims. of B & D;
5	Def. 5, V.	$\therefore$ K $>$ = or $<$ G, so L $>$ = or $<$ H.
6	C.	But K, L are equims of E, F, & G, H any of B, D;
7	Def. 5, V.	$\therefore$ E : B = F : D.
8	*Sim.*	In the same way, A : G = C : H

Cor. 2. $\because$ in Dem. 8, Pr. 4, V., if K $>$ = or $<$ M, L $>$ = or $<$ N;

$\therefore$ if M $>$ = or $<$ K, N $>$ = or $<$ L. Hence G : E = H : F. Therefore, *if four magnitudes are proportionals, they will be proportional* by inversion.

N.B.—This Cor. is not in its proper place; it correctly forms Prop. B, V.

Cor. 3. If A : B = C : D,—and *if any like parts* of A and C *be taken*, as $\frac{A}{2}$, $\frac{C}{2}$, *and also any like parts* of B and D, as $\frac{B}{3}$, $\frac{D}{3}$, *these like parts will also be proportional*; i. e., $\frac{A}{2} : \frac{C}{2} = \frac{B}{3} : \frac{D}{3}$.

Alg. & Arith. Hyp.—Let $a = 2$, $b = 5$, $c = 4$, $d = 10$, $m = 2$ & $n = 3$.

Alg.—$\because$ $a : b = c : d$, then $ma : nb = mc : nd$. For, 3, V. equims. of ma and mc are equims. of a & c; and equims of nb, nd also equims. of b & d.

But, Def. 5, V., a & c $<$ = or $>$ b & d, $\therefore$ equims. of a & c $<$ = or $>$ b & d, and also $<$ = or $>$ equims. of b & d.
Hence, Def. 5, V. $ma : nb = mc : nd$.

Arith. $\because$ 2 : 5 = 4 : 10, then $2 \times 2 : 3 \times 5 = 2 \times 4 : 3 \times 10$. Now equims. of 4 & 8 are equims. of 2 & 4; and equims of 15 & 30 also equims. of 5 & 10.

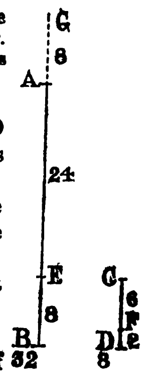
But, Def. 5, V. $\because$ 2 & 4 $<$ = or $>$ 5 & 10 $\therefore$ equims of 2 & 4 $<$ = or $>$ 5 & 10, and also $<$ = $>$ equims. of 5 & 10. Hence, Def. 5, V. 4 : 15 = 8 : 30.

Appl.—From Cor. 3, arises the rule in simple proportion in arithmetic, of dividing the 1st and 2nd terms by any common measure, and using the *resulting* instead of the original numbers.

Prop. V.—Theor.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder is the same multiple of the remainder, that the whole is of the whole.

Con.—Pst. 1, V. Dem. 1, V. Ax. 1, V. Equimultiples of the same or of equal magnitudes are equal to one another. Ax. 3. 1.—If equals be taken from equals the remainders are equal.



E. 1	Hyp. 1	Let A B be the same *m* of C D that A E, taken from A B, is of C F, taken from C D;
2	Conc.	then the rem. E B is the same *m* of rem. F D, as the whole A B of the whole C D.
C	Pst. 1, V.	Take A G same *m* of F D, that A E is of C F.
D. 1	C.	∵ A G, A E are equims. of F D and C F,
2	1, V.	∴ A G, A E, *i. e.*, E G, same *m* of C F, F D, *i.e.*, C D, that A E is of C F.
3	H.	but A E same *m* of C F, that A B is of C D;
4	1, V.	∴ E G same *m* of C D that A B is of C D,
5	Ax. 1, V.	∴ E G = A B;
6	*Sub* & Ax. 3, I.	from each take A E; ∴ rem. A G = rem. E B.
7	C.	∵ A E same *m* of C E that A G is of F D,
8	D. 6	and A G = E B;
9	1, V.	∴ A E same *m* of C F that E B is of F D;
10	H.	But A E same *m* of C F that A B is of C D;
11	1, V.	∴ E B same *m* of F D that A B is of C D.
12	Conc.	*Therefore if one magnitude be the same, &c.*

Q. E D.

Alg. & Arith. Hyp.—Let A = 32, B = 8, C (a part of A) = 24, and D (a part of B) = 6, $m = 4$.

Alg.	Let $A = mB$.	*Arith.*	Let $32 = 4 \times 8$.
	$C = mD$;		$24 = 4 \times 6$.
Subt.	$A - C = m(B - D)$,	Subt.	$8 = 4(8 - 6)$;
Thus	$A - C$ is m times $(B - D$,	Thus	8 is 4 times 2;
as	A is m ,, B.	as	32 was 4 ,, 8.

So the rem. is the same m of the rem. as the whole is of the whole.

Or, Let $A - B = D$; to both sides add B,—then $A = B + D$;
∴ (1, V.) $mA = mB + mD$;
Subtract mB, and $mA - mB = mD$;
but $D = A - B$; ∴ $mA - mB = m(A - B)$;
Thus the rem. is the same m of the rem. that mA is of A.

SCH. "*If from a multiple of a magnitude by any number, a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude that the difference of the numbers is of unity.*

E. 1	Hyp.	Let mA, nA be mults. of A, m being $> n$;
2	Conc.	then $mA - nA = (m - n)A$.
C.	*Sum.*	Let $m - n = q$; then $m = n + q$.
D. 1	C & 2, V.	Here $mA = nA + qA$;
2	*Sub.*	from both take nA; then $mA - nA = qA$;
3	Conc.	∴ $mA - nA = (m - n)A$.

COR. When the difference of the two numbers is equal to unity, or $m - n = 1$, then $mA = nA = A$; or $2A - A = A$.

PROP. VI.—THEOR.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

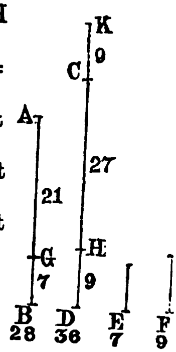
CON. Pst. 2, I.—3, I. From the gr. of two lines to cut off a part equal to the less. Pst. 1, V.

DEM. Ax. 1, V.—Ax. 3, I. Ax. 1, I.—2, V.

E. 1	Hyp. 1.	Let A B, C D be equims. of E F;
2	,, 2.	and let A G taken from A B be an equim. of E,
3	,, 3.	and C H from C D an equim. of F;
4	Conc.	then rems. G B, H D either = E F, or are equims. of them.

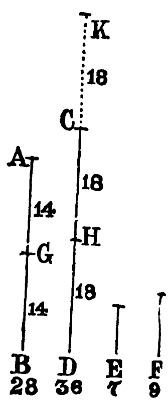
CASE. I.—Let *G B = E, then H D shall equal F.*

C.	Pst. 2 I & 3 I	Make C K = F.
D. 1	H. 2	∵ A G same *m* of E, that C H is of F;
2	H. & C.	and ∵ G B = E, and C K = F;
3	Conc.	∴ A B same *m* of E, that K H is of F.
4	H.	But A B same *m* of E, that C D is of F.
5	Conc.	∴ K H same *m* of F that C D is of F.
6	Ax. 1 V.	∴ K H = C D,
7	*Sub* Ax. 3 I.	take away C H, ∴ rem. K C = rem. H D;
8	C. Ax. 1 I.	but K C = F, ∴ H D = F.



CASE II.—Let G B be a *m* of E, then H D same *m* of F.

C.	Pst. 1, V.	Of F take C K the same *m* that G B is of E.
D. 1	H.	∵ A G same *m* of E that C H is of F.
2	C.	and G B same *m* of E that K H is of F;
3	2, V.	∴ A B same *m* of E that K H is of F.
4	H.	But A B same *m* of E that C D is of F,
5	Conc.	∴ K H same *m* of F that C D is of F,
6	Ax. 1, V.	∴ K H = C D.
7	*Sub.* Ax. 3, I.	Take away C H, ∴ rem. KC, = rem. HD.
8	C.	And ∴ G B same *m* of E that K C is of F,
9	D. 7	and K C = H D;
10	Conc.	∴ H D same *m* of F that G B is of E.
17	Rec.	*If*, therefore, *two magnitudes, &c.* Q. E. D.



Alg. & Arith. Hyp.—Let $m > n$ express any integers, as 4 and 3; $A = 28$, $B = 36$, $C = 7$, $D = 9$.

Alg.	*Arith.*
Alg.—Let A, B be equims. of C & D	*Arith.*4 Let 28 &36 be equims. of 7 &9,
$A = m\,C$, $\quad B = m\,D$,	$28 = \ \times 7$; $\quad 36 = 4 \times 9$
$A > n\,C$, $\quad B > n\,D$;	$28 > 3 \times 7$; $\quad 36 > 3 \times 9$;
Subt. $A - n\,C = m\,C - n\,C = (m-n)C$;	Subt. $28 - 21 = 4 \times 7 - 3 \times 7 = (4-3)\,7$.
$B - n\,D = mD - nD = (m-n)D$.	$36 - 27 = 4 \times 9 - 3 \times 9 = (4-3)\,9$.
Suppose $m - n = 1$.	Here $4 - 3 = 1$.
1°. $A - n\,C = C$,	1°. $28 - 21 = 7$.
$B - n\,D = D$,	$36 - 27 = 9$,
2°. *Or*, equims of C & D	2°. *Or*, equims. of 7 & 9.
$(m-n)$ C and $(m-n)$ D.	as $(5-2)\,7$, and $(5-2)\,9$.

SCH.—The six preceding Propositions are chiefly useful for establishing, by the method of Equimultiples, the Propositions which follow. When this method is not employed some have adopted the Postulate,—*Three Magnitudes, A, B, C, being given, let it be granted that there is a 4th magnitude, we may call it x, to which C has the same ratio, as A to B*; *i. e.* $A : B = C : x$.

PROP. A,—THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then if the first be greater than the second the third is greater than the fourth; and if equal, equal; if less, less.

CON.—Pst. 1, V. DEM.—Def. 5, V.

E.1	Hyp.	Let $A : B = C : D$,
2	Conc.	then if $A >\ =$ or $< B$, $C >\ =$ or $< D$.
C	Pst. 1, V.	Take any equims., as 2 A, 2 B, 2 C, 2 D.
D.1	H.	$\because A : B = C : D$,
2	C.	and of the 1st and 3rd equims. 2 A, 2 C are taken, and of the 2nd and 4th equims. 2 B, 2 D;
3	Def. 5, V.	$\therefore$ 2 C is $>\ =$ or $<$ 2 D, as 2 A is $>\ =$ or $<$ 2 B.
4	,,	But 2 C is $>\ =$ or $<$ 2 D as C is $>\ =$ or $<$ D;
5	*Sim.*	and so, 2 A is $>\ =$ or $<$ 2 B, as A is $>\ =$ or $<$ B;
6	Conc.	$\therefore$ C, is $>\ =$ or $<$ D, as A is $>\ =$ or $<$ B. Q. E. D.

Or, more briefly;

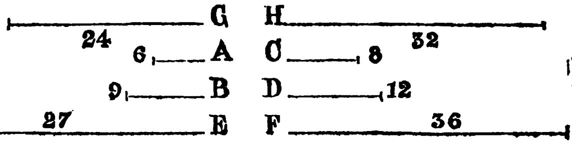
D.1	Def. 5, V.	If 2 A $>$ 2 B, 2 C $>$ 2 D;
2	„	but if A $>$ B, 2 A $>$ 2 B;
3	Conc.	$\therefore$ 2 C $>$ 2 D, and $\therefore$ C $>$ D.
4	*Sim.*	So if A $=$ or $<$ B, C $=$ or $<$ D. Q. E. D.

USE.—SIMSON added this and the next three Propositions. Prop. A is required for the demonstration of 25, V; 21, VI; 34, XI, and 15, XII, and is often employed by Geometers.

PROP. B.—THEOR.

Invertendo, by inverting.—*If four magnitudes are proportionals, they are proportionals also when taken* inversely.

CON. Pst. 1, V. DEM. Def. 5, V.



E. 1	Hyp.	Let A : B $=$ C : D;
2	Conc.	then B : A $=$ D : C.
C. 1	Pst. 1, V.	Of B,D, take any equims. E,F;
2	„	and of A,C, any equims. G,H.

First.—Let E $>$ G. *i. e.*, G $<$ E.

D. 1	H.	$\because$ A : B $=$ C : D,
2	C. 2.	and of the 1st A, and 3rd C, are equims. G & H,
3	C. 1.	and of the 2nd B, and 4th D, „ E & F;
4	Def. 5. V.	$\therefore$ H $>$ $=$ or $<$ F, as G $>$ $=$ or $<$ E;
5	Conc.	If $\therefore$ E $>$ G, F is $>$ E.

Second.—So, if E $=$ or $<$ G, F $=$ $<$ H.

6	C.	But E,F any equims. of B,D; & G,H of A,C;
7	Def. 5, V.	$\therefore$ B : A $=$ D : C.
8	Rec.	Therefore, *if four magnitudes, &c.* Q. E. D.

Alg. & Arith. Hyp. Let $A = 6$, $B = 9$, $C = 8$; $D = 12$; $m = 4$, $n = 3$.

Alg. Let m A & mC be equims. of A & C.
nB & nD „ B & D;
$\because$ A : B = C : D
Def. 5, V. $\therefore$ mA, mC $>$ = or $<$ nB, nD.
If mA, mC $>$ nB, nD;
nB, nD $<$ mA, mC;
If mA, mC $<$ Bn, nD D;
nB, nD $>$ mA, mC;
$\therefore$ Any equims. of B, D $>$ = or $<$ A,D.
Def. 5, V. $\therefore$ B : A = D : C.

Arith. Let 4×6, & 4×8 be equims. of 6×8,
3×9, & $3,\times12$ „ of 9 & 12.
$\because$ 6 : 9 = 8 : 12.
$\therefore$ 24, 32 $<$ 27 & 36.
i. e. 27 & 36 $>$ 24 & 32.
$\therefore$ equims. of 9 & 12 $>$ those of 6 & 8.
$\therefore$ 9 : 6 = 12 : 8.

Sch. 1. The Proposition may be stated,—"The reciprocals of equal ratios are equal to one another."

2. By an inaccuracy Prop. B has been placed by some as a Corollary to Pr 4. V.

Prop. C.—Theor.

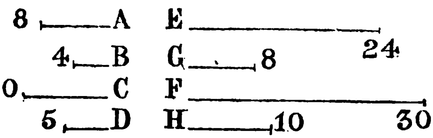
If the first be the same multiple of the second, or the same part (i. e. measure, or measure submultiple) *of it that the third is of the fourth, the first is to the second, as the third is to the fourth.*

Con. Pst. 1 V. Dem. 3 V.—Def. 5 V.—Def. 1 V.—Def. 2 V., P. B. V.

There are *two* Cases of this Prop. according as A and C are multiples or parts of B and D.

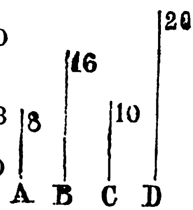
Case I.—*Let A & C be multiples of B & D.*

E.	1	Hyp.	Let A 1st be same m of B 2nd, that C 3rd is of D 4th,
	2	Conc.	then A : B = C : D.
C.	1	Pst. 1. V.	Take of A, C any equims. E, F;
	2	„	and of B, D any equims. G, H.



D. 1	H. & C.	∵ A, C equims. of B, D and E, F of A, C;
2	3, V.	∴ E, F equims. of B, D.
3	C.	But G, H equims. of B, D;
4	Conc.	∴ if E $>$ *m* of B than G is of B, F is a $>$ *m* of D than H of D;
5	Remk.	*i. e.* if E $>$ G, F $>$ H.
6	*Sim.*	So, if E $=$ or $<$ G, F $=$ or $<$ H.
7	C.	But E, F are equims. of A, C, and G, H of B, D;
8	Def. 5, V.	∴ A : B = C : D. Q. E. D.

CASE II. *Let A & C be parts, or sub-multiples, of B & D.*



E. 1	H.	Let A the 1st, be the same part of B the 2nd, that C the 3rd, is of D the 4th;
2	Conc.	then A : B = C : D.
D. 1	Def. 1, V.	∵ A the same part of B that C is of D,
2	Def. 2, V.	∴ the same *m* of A that D is of C;
3	Case 1.	∴ B : A = D : C.
4	B. V. *Invert.*	∴ A : B = C : D. Q. E. D
5	Rec.	Therefore, *if the first be the same multiple, &c.*

Alg. & Arith. Hyp. Take A = 2 B = 8, and C = 2 D = 10; $n = 3$ & $p = 4$.

Alg. Take *n* A, *n* C any equims. of A, C;
p B, *p* D „ of B, D,
Then ∵ A same *m* of B, that C is of D;
and *n* A „ „ of A, „ *n* C is of C;
∴ *n* A „ „ of B, „ *n* C is of D,
i. e. *n* A, *n* C are equims of B, D.
But *p* B, *p* D „ „ of B, D;
∴ If *n* A $>$ *m* of B, than *p* B is of B,
n C $>$ *m* of D, „ *p* D of D;
i. e., if *n*A $>$ *p*B, *n*C $>$ *p*D.
So, if *n* A = or $<$ *p*B, *n*C = or $<$ *p*D.
But *n*A, *n*C equims. of A, C,
*p*B, pD, „ of B, D,
∴ by Def. 5. V. A : B = C : D.

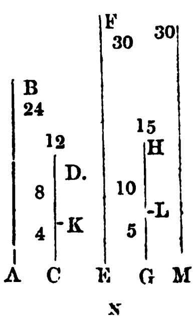
Arith. 3×8, & 3×10 are equims. of 8 & 10;
4×4 & 4×5 are equims. of 4 & 5;
Then ∵ 8 is the same *m* of 4 that 10 is of 5;
24 is the same *m* of 8 that 30 is of 10;
∴ 24 is the same *m* of 4 that 30 is of 5;
i. e., 24 & 30 are equims. of 4 & 5.
But 16 & 20 „ „ 4 & 5;
∴ If 24 > *m* of 4, than 16 is of 4,
30 > *m* of 5, „ 20 of 5,
i. *e.*, if 24 > 16, 30 > 20.
So, if 27 = or < 16, 30 = or < 20.
But 24 & 30 are equims. of 8 & 10,
16 & 20 „ 4 & 5.
∴ by Def. 5, V. $8 : 4 = 10 : 5$.

SCH.—The 7th, 8th, 9th, and 10th books of EUCLID's Entire Work treat of Arithmetic and the doctrine of Incommensurables; and the 20th Def. Book VII., gives a definition of quantities which are proportional; but "most of the commentators" says SIMSON, and POTTS repeats the words, "judge it difficult to prove that four magnitudes which are proportionals according to the 20th def. of the 7th book; are also proportionals according to the 5th def. of the 5th book. The Demonstration, however, is as follows; from SIMSON's Notes, page 317.

I°. *As to four magnitudes in proportion according to Def. 5, V.*

CASE 1.—Let A, B, C, D be four *Ms*, such that $A = m\,B$, or $\frac{B}{p}$; and $C = m\,D$, or $\frac{D}{p}$, ; then, by Pr. C, bk. V; $A : B = C : D$.

CASE 2.—In A B let there be the same parts of C D as there are of G H in E F, then also $AB : CD = EF : GH$; For



C. 1	Pst.2,V.	Let C K be of C D the same *pt.* that G L is of G H,
2	„	and A B same *m* of C K that E F is of G L;
D.1	Pr. C,V.	∴ $AB : CK = EF : GL$;
2	C.	And ∵ C D, G H are equims. of C K, G L, the 2nd and 4th,
3	Cor.4,V.	∴ $AB : CD = EF : GH$

II°.—And, *if four magnitudes are proportionals according to the 5th def. of Book V, they are also proportionals according to the 20th def. of Book VII.*

CASE 1.—Let A : B = C : D.
then, by Pr. D, book V, if A is any *m*, or *pt* of B, C is the same *m* or *pt* of D.

CASE 2.—Let A B : C D = E F : G H, as in the foregoing figure: then if A B = any *pt* of C D, E F = the same *pt* of G A.

C	1	Pst. 2, V.	Take C K a *pt* of C D, and G L the same *pt* of G H;
	2	Pst. 1, V.	and let A B be a *m* of C K, and E F the same *m* of G L;
	3	Pst. 2, V.	take M the same *m* of G L that A B is of C K;
D	1	Pr. C, V.	∵ A B : C K = M : G L,
	2	C	and C D, G H = *m* C K, *m* G L:
	3	Cor. 4, V.	∴ A B : C D = M : G H;
	4	H.	And A B : C D = E F : G H,
	5	9, V.	∴ M = E F
	6	Conc.	∴ E F the same *m* of G L that A B is of C K.

"This is the method by which Simson shows that the Geometrical definition of proportion is a consequence of the Arithmetical definition, and conversely."

"It may, however, be shewn by employing the equation $\frac{a}{b} = \frac{c}{d}$, and taking ma, mc any equims. of a and c, the first and third, and nb, nd, any equims. of b and d, the second and fourth."—POTTS.

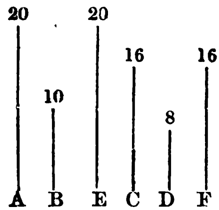
USE.—Prop. C, bk. V., is often made use of, and "is necessary to the 4th and 6th propositions of the 10th Book."

PROP. D.—THEOR.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second, the third is the same multiple or the same part of the fourth.

CON.—Pst. 1, V. DEM.—Cor. 4, V. A, V, B, V.

E.1	Hyp.	Let $A : B = C : D$; and $A = m\,B$, or $\frac{B}{p}$;
2	Conc.	then $C = m\,D$, or $\frac{D}{p}$.

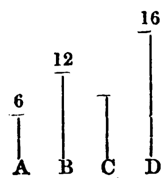


CASE 1.—Let $A = m\ B$; then $C = m\ D$.

C.1	Pst. 1, V.	Take $E = A$, and F the same *m* of D that A or E is of B.
D.1	H.	$\because A : B = C : D$,
2	C.	and of B, D, 2nd and 4th, the equims are E and F;
3	Cor. 4, V.	$\therefore A : E = C : F$.
4	C. A. V.	But $A = E$, $\therefore C = E$;
5	C.	And F is the same *m* of D that A is of B;
6	Conc.	$\therefore$ C ,, *m* of D that A is of B.
		Q. E. D.

CASE 2.—Let $A = \frac{B}{p}$, then $C = \frac{D}{p}$.

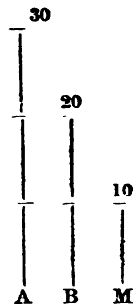
D. 1	H. & B, V.	$\because A : B = C : D$, $\therefore$ *invert.* $B : A = D : C$;
		but $A = \frac{B}{p}$, $\therefore$ B is a *m* of A;
2	H.	and $\therefore$ D the same *m* of C,
3	Case 1.	*i. e.*, $C = \frac{D}{p}$, or same *pt.* of D, that A is of C.
4	Remk.	Therefore, *if the first be to to the second, &c.*
5	Rec.	Q. E. D.



N.B.—For sake of preserving the same numbering we insert here as Prop. 7 A, the 7th Prop. of CHAMBERS' *Additional Fifth Book.*

PROP. 7 A.---THEOR.

The ratio of two lines is the same as that of the numbers which express the number of times that any third line is contained in them respectively.



E. 1	Hyp.	Let A=30, and B=20, be two lines, and M = 10 a third;
2	"	and let $M = \frac{A}{a}$, and $\frac{B}{b}$,
3	Conc.	then $\frac{A}{B} = \frac{a}{b}$; or A : B = $a : b$.

CASE.—Let a and b be integers, and a : = 3, b = 2.

D. 1	H.	∵ A = 3M, and B = 2M:
2	Conc.	∴ A contains $1\frac{1}{2}$B, & a, $1\frac{1}{2}b$.
3	H.	and ∴ A : B = $a : b$, or 30 : 20 = 3 : 2.
4	*Sim.*	So we conclude whatever be the integer values of a and b;
5	Conc.	$\therefore \frac{a}{b} = \frac{aM}{bM} = \frac{A}{B}$, i. e. $\frac{3}{2} = \frac{3 \times 10}{2 \times 10} = \frac{30}{20}$.

If being integers, $a = mq$ and $b = np$, then $\frac{a}{b} = \frac{mq}{np} = \frac{mq\,M}{np\,M}$.

Or, $6 = 3 \times 2$, and $8 = 2 \times 4$;
then $\frac{6}{8} = \frac{3 \times 2}{2 \times 4} = \frac{3 \times 2 \times 10}{2 \times 4 \times 10} = \frac{60}{80}$.

CASE 2. Let $a = \frac{m}{n}$, and $b = \frac{p}{q}$; or $3 = \frac{6}{2}$, and $2 = \frac{8}{4}$.

D. 1	Hyp.	∴ M′ is a *subm.* of M by nq, or M = nq M′, i. e., $10 = 2 \times 4 \times 1\frac{1}{4}$;
2	Case 1.	$\therefore \frac{a}{b} = \frac{m}{n} \times \frac{q}{p} = \frac{mq}{np} = \frac{mq\,M'}{np\,M'}$, i. e., $\frac{3}{2} = \frac{6}{2} \times \frac{4}{8} = \frac{24 \times 1\frac{1}{4}}{16 \times 1\frac{1}{4}} = \frac{30}{20}$.
3		But $mq\,M' = \frac{mnq\,M'}{n} = \frac{m}{n} \times nq\,M' = \frac{m}{n}M$; i. e. $24 \times 1\frac{1}{4} = \frac{6}{2} \times 10 = 30$.
4	*Sim.*	So $np\,M' = \frac{p}{q}M$, i. e., $2 \times 8 \times 1\frac{1}{4} = \frac{8}{4} \times 10 = 20$;
5	Conc.	$\therefore a : b = \frac{m}{n}M : \frac{p}{q}M$, i. e., $3 : 2 = \frac{6}{2} \times 10 : \frac{8}{4} \times 10$; or $a : b = A : B$; $\frac{3}{2} = \frac{30}{20}$.

Case 3. Let a be interminate and b terminate.

	Sup.	If $\frac{A}{B} \neq \frac{a}{b}$, let $\frac{A}{B} = \frac{a'}{b'}$; and let $a' > a$, and a'' be a terminate no. between a & a'.
D. 1	Case 1 & 2.	$\because \frac{a''}{b} = \frac{a''M}{b\ M}$, and $a''M > aM$, or A;
2		$\therefore \frac{a''M}{b\ M} > \frac{A}{B}$; $\frac{a''}{b} > \frac{a'}{b}$, and $\therefore a'' > a'$
3	H.	But a'' also $< a'$, an impossibility, $\therefore a' \not> a$.
4	*Sim.*	So $a' \not< a$, $\therefore \frac{A}{B} = \frac{a}{b}$

Case 4. Let a be terminate and b interminate.

D.	*Sim.*	By a demonstration similar to that of Case 3.

Case 5. Let a and b be both interminate.

	Sup.	If $\frac{A}{B} \neq \frac{a}{b}$, let $\frac{A}{B} = \frac{a'}{b}$; let $a' > a$ & a'' be a terminate no. between a & a'
D. 1	Case 4.	Then $\frac{a''}{b} = \frac{a''M}{b\ M}$.
2	*Sim.*	The rest of the dem. as in Case 3.
3		$\therefore \frac{A}{B} = \frac{a}{b}$

Prop. VII.—Theor.

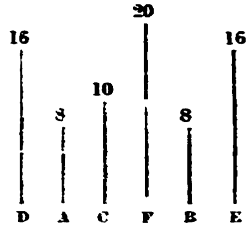
Equal magnitudes have the same ratio to the same magnitude, and conversely, *the same has the same ratio to equal magnitudes.*

Con. Pst. 1, V. A given M. may be so increased that any required m of it may be taken.

Dem. Ax. 1, V. Equims, of the same or of equal Ms. or equims. are equal to one another.

Def. 5, V. The first of four magnitudes is said to have the *same ratio*, &c.

E. 1	Hyp. 1	Let A = B, & C be a 3rd M. of the same kind;
2	Conc. 1 & 2.	then A : C = B : C; and C : A = C : B.
C. 1	Pst. 1, V.	Of A, B, take any equims. D, E;
2	Pst. 1, V.	and of C, take any equim. F.



First. A and B shall each have the same ratio to C.

D. 1	C.	∵ D & E are equims. of A & B;
2	H. Ax. 1, V.	and A = B, ∴ D = E;
3		∴ if D > = or > F, E > = or > F;
4	C. 1 & 2.	but D, E are equims. of A, B, and F is a *m* of C;
5	Def 5, V.	∴ A : C = B : C.

Second. C shall have the same ratio to A that it has to B.

D. 1	*Sim.*	As before, D = E;
2		∴ if F > = or > D, F > = or > E;
3	C. 2 & 1.	but F any *m* of C, and D, E any *m* of A, B;
4	Def. 4, V.	∴ C : A = C : B.
5	Rec.	Therefore, *equal magnitudes have the same ratio, &c.* Q. E. D.

Cor.—If a ratio A : C which is compounded of two ratios, A : B, and B : C, be a ratio of equality, one of these must be the inverse, or reciprocal of the other; *i. e.*, A : B is the inverse, or reciprocal of B : C.

Sch.—The second part, C : A = C : B, follows from the corollary of Pr. 4, Bk. V.

Prop. VIII.—Theor.

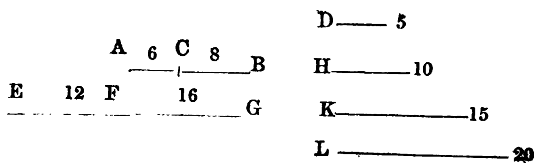
Of two unequal magnitudes, the greater has a greater ratio to another magnitude than the less has; and conversely, *the same magnitude has a greater ratio to the less of two other magnitudes, than it has to the greater.*

Con. Pst. 1, V. Dem. 1, V. Def. 7, V.—When of the equims. of four Ms., (taken as in the fifth def.) the *m* of the 1st is > that of the 2nd, but the *m* of the 3rd ⯒ than the *m* of the 4th; then the first is said to have to the 2nd a greater ratio than the 3rd M has to the 4th; and on the contrary, the 3rd is said to have to the fourth a less ratio than the 1st has to the second.

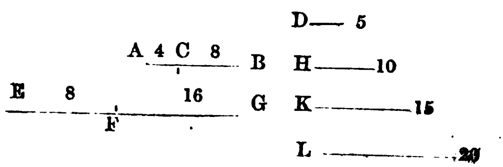
E.	1	Hyp.	Let A B > B C, and D any other M;
	2	Conc. 1.	then A B : D > B C : D;
	3	„ 2.	and D : B C > D : A B.

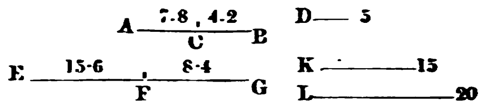
First.—A B : D > B C : D.

C.	1	*Sup.* 1.	If of the two *Ms.*, A C, C B, the *not gr.* of the two be *not less* than D;
	2	Pst. 1, V.	take E F = 2 A C, and F G, = 2 C B,



C.	3	*Sup.* 2.	But if of the two *Ms.*, the *not gr.* M be < D;
	4	Pst. 1, V	take of A C, C B, equims. E F, F G each > D.





C. 5	Pst. 1, V.	In all cases take H = 2 D, K = 3 D, &c., till the *m* of D be that which is first > F G.
6	Pst. 1, V.	Let L, a *m* of D, be found > F G;
7	"	and K the *m* of D next less to L; *i. e.*, FG ≮ K.
D. 1	C.	∵ L is the *m* of D which first is > FG;
2		∴ K, the *m* of D next preceding, is ≯ F G, *i. e.*, F G ≮ K.
3	C.	And ∵ EF same *m* of AC, that FG is of CB;
4	1, V.	∴ FG the same *m* of CB that E G is of AB;
5	Remk.	*i. e.*, EG and FG are equims. of AB and CB.
6	D. 2, & C.	And ∵ FG ≮ K, and EF > D;
7		∴ the whole E G = FG + FE > K + D;
8	C.	but K + D = L, ∴ EG > L.
9	C. D. 5.	But FG ≯ L, and EG, FG equims of AB, CB;
10	C.	and L is a *m* of D;
11	Def. 7. V.	∴ AB : D > BC : D. Q. E. D.

Second.—D : BC > D : AB.

C.	*Sim.*	As in the *first* part.
D. 1	*Sim.*	As before, L > FG but ≯ E G;
2	C.	and L a *m* of D;
3	D. 5. pt. 1.	also FG, EG were proved equims of CB, AB;
4	Def. 7, V.	∴ D : BC > D : AB.
5	Rec.	∴ *Of two unequal magnitudes, &c.*

Q. E. D.

Otherwise. Let A + B = 5 + 4, and A = 5, be two unequal Ms; and C = 10 a third M; then A + B : C > A : C; *i. e.*, (5 + 4) : 10 > 5 : 10.

Alg. Take mA, mB, each $>$ C;
nC the least mult. $> mA + mB$.
Hence $(n-1)\ C < mA + mB$, or $m\ (A + B)$;
$m\ (A + B) > (n-1)\ C$, or $nC - C$;
$\because\ n\ C > (mA + mB)$, and $C < mB$,
$\therefore\ n\,C - C > m\,A$, or $m\,A < (n-1)\ C$.
Hence $m\ (A + B)$ a m of $A + B > (n-1)\ C$, a m of C.
but mA, *is* a m of A $\not> (n-1)\ C$, a m of C,
$\therefore$ by Def. 7, V. $A + B : C > A : C$.

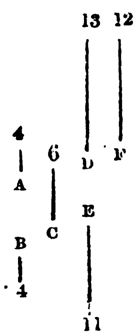
Arith. 3×5, and 3×4, each > 10;
3×10 least $m > (3 \times 5) + (3 \times 4)$.
Hence $(3-1)\ 10 < (3 \times 5) + (3 \times 4)$ or $3\ (5+4)$.
$3\ (5+4) > (3-1)\ 10$, or $(3 \times 10) - 10$.
$\because\ 3 \times 10 > (3 \times 5) + (3 \times 4)$, and $10 < 3 \times 4$.
$\therefore\ (3 \times 10) - 10 > 3 \times 5$, or $3 \times 5 < (3-1)\ 10$.
Hence $3\ (5+4) > (3-1)\ 10$.
but $3 \times 5 \not> (3-1)\ 10$,
$\therefore\ 5 + 4 : 10 > 5 : 10$.

Prop. IX.—Theor.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

Con. Pst. 1, V. Dem. 8, V.—Def. 7, V.—Def. 5, V.

E.	1	Hyp. 1.	Let $A : C = B : C$;
	2	„ 2.	or $C : A = C : B$;
	3	Conc.	then $A = B$.



First. Let $A : C = B : C$, then $A = B$.

D.	1	*Sup.*	If $A \neq B$, let $A > B$;
	2	8, V.	then the gr. M, $A : C >$ the less M, $B : C$.
	3	Def. 7, V.	Now equims. of A, B, mA, mB, and a m of C, $(n-1)\ C$, may be taken, so that $mA >$, and $mB \not> (n-1)\ C$;

	4	Pst. 1, V.	Of A & B take equims. D,E; and of C a m, F, so that $D > F$, but $E \ngtr F$;
	5	Hyp.	then $\because A : C = B : C$;
	6	D. 4.	and of A & B equims. D, E, and of C a m, F, and $D > F$;
	7	Def. 5, V.	$\therefore E > F$;
	8	D. 4.	But E also $\ngtr F$;—two things, of which one is impossible;
	9	Conc.	$\therefore$ A not $\neq$ B, *i. e.* $A = B$.

Second. Let $C : A = C : B$,—then $A = B$.

D.	1	*Sup.*	If $A \neq B$, let $A > B$.
	2	8, V.	then C : B the less $M >$ C : A the gr. M.
	3	Def. 7, V.	$\because$ there may be taken a m of C the 1st and 3rd, and equims. of A & B the 2nd and 4th; so that m of $C > m$B, but $\ngtr m$ A;
	4	Pst. 1, V.	Take F, a m of C, and E, D, equims. of B, A, so that $F > E$, but $\ngtr$ D.
	5	Hyp.	And $\because C : B = C : A$,
	6	D. 4.	and F, m of $C > E$, m of B;
	7	Def. 5. V.	$\therefore$ F, a m of $C >$ D, a m of A;
	8	H. Conc.	but F also $\ngtr$ D; an impossibility, $\therefore A = B$.
	9	Rec.	$\therefore$ *Magnitudes which have the same ratio, &c.* Q. E. D.

Otherwise. Hyp.—Let $A : C = B : C$, or $C : A = C : B$; then $A = B$; or let $4 : 6 = 4 : 6$ and $6 : 4 = 6 : 4$, then $4 = 4$

Case I.—If $A \neq B$, let $A > B$.
Assume, (8, V.) $m\,A > n\,C$, but $m\,B \ngtr n\,C$:
$\because A : C = B : C$, $\therefore m\,A, m\,B >= \text{or} < n\,C$;
but $m\,A > n\,C$, and $m\,B < n\,C$, an absurdity;
$\therefore$ A not $\neq$ B, *i. e.*, $A = B$.

Case II.—If $C : A = B : A$, then $A = B$.
Invertendo. $A : C = B : C$; $\therefore$ by Case 1, $A = B$.

Cor.—*A ratio compounded of two ratios, of which one is the reciprocal of the other, is a ratio of equality.*

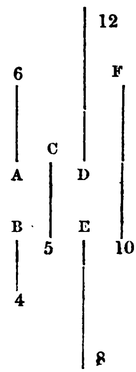
For in A, B, C, magnitudes of the same kind,—if $B : C = B : A$, $A = C$;
i. e., the ratio A : C compounded of A : B and of B : C, one the reciprocal of the other, is a ratio of equality. Note Def. 3, V. Def. A, V.

Prop. X.—Theor.

That magnitude which has a greater ratio than another has unto the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

Con.—Pst. 1, V. Dem. Def. 7, V. Ax. 4, V. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

E.	1	Hyp. 1.	Let A : C $>$ B : C;
	2	Conc. 1.	then A $>$ B.
	3	Hyp. 2.	Let C : B $>$ C : A;
	4	Conc. 2.	then B $<$ A.



Case I.—Let A : C $>$ B : C, then A $>$ B.

D.	1	Hyp. 1.	$\because$ A : C $>$ B ; C;
	2	Def. 7, V.	$\therefore$ of A 1st and B 3rd can be taken some equims., and of C the 2nd and 4th a *m* so that *m* A $>$ *m* C and *m* B $\not>$ *m* C.
	3	Pst. 1, V.	Take D, E, such equims., of A, B, and F a *m* of C, so that D $>$ F but E $\not>$ F;
	4		$\therefore$ D $>$ E.
	5	D. 3.	And $\therefore$ D and E are equims. of A and B,
	6	Ax. 4, V.	and if the 1st $<$ the 2nd, the *m* of the 1st $<$ the same *m* of the 2nd;
	7	D. 4. Conc	and D $>$ E; $\therefore$ A $>$ B.

Case II.—Let C : B $>$ C : A, then B $<$ A.

D.	1	Hyp.	For $\therefore$ C : B $>$ C : A;
	2		$\therefore$ of C 1st and 3rd can be taken some *m*, and of B, A, 2nd and 4th equims, so that *n* C $>$ *m* B, but $\not>$ *m* A.
	3	Pst. 1. V.	Of C take such *m*, F, and of B, A equims. E, D, so that F $>$ E but $\not>$ D;
	4		$\therefore$ E $<$ D.
	5	D. 3, 4.	and $\because$ E, D are equims. of B, A, and E $<$ D;
	6	Ax. 4. V.	$\therefore$ B $<$ A.
	7	Rec.	Therefore, *that Magnitude which has a greater &c.*

Q. E. D.

Otherwise. Hyp. 1°. Let A : C > B : A, then A > B ; or 6 : 5 > 4 : 5, then 6 > 4.

2°. Let C : B > C : A, then B < A ; or 5 : 4 > 5 : 6, then 4 < 6.

Alg. 1°. Def. 7, V. Take m A, > n C, and m B < n C.
∴ m A > m B, and Ax. 4, V. A > B.

2°. Def. 7, V. Take n C > m B, and n C < m A.
∵ m B < n C, and m A > n C ;
∴ m B < m A, and ∴ B < A.

Arith. 1°. 3 × 6 > 3 × 5 ; but 3 × 4 < 3 × 5.
∴ 3 × 6 > 3 × 4, and ∴ 6 > 4.

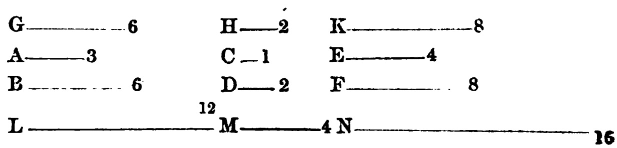
2°. 3 × 5 > 3 × 4, but 3 × 5 < 3 × 6.
∵ 3 × 4 < and 3 × 6 >, 3 × 5.
∴ 3 × 4 < 3 × 6 ; ∴ 4 < 6.

Prop. XI.—Theor.

Ratios that are the same to the same ratio, are the same to one another.

Con.—Pst. 1, V. Dem. Def. 5, V. Definition of Proportion.

E.	1	Hyp.	Let A : B = C : D, & C : D = E : F ;
	2	Conc.	then A : B = E : F.



C.	1	Pst. 1, V.	Of A, C, E, take any equims. G, H, K ;
	2	,,	and of B, D, E, take any equims. L, M, N.
D.	1	Hyp.	∵ A : B = C ; D,
	2	,,	and G, H are equims. of A, C ; L, M equims. of B, D ;
	3	Def. 5, V.	∴ if G > L, H > M ; if =, = and if less, less.
	4	Hyp.	Again, ∵ C : D = E : F,
	5	,,	and H, K are equims. of C, E ; M, N, of D, F ;
	6	Def. 5, V.	∴ if H > = or < M, K > = or < N.
	7	D, 3.	but if G > L, H > M, if =, =, and if less, less ;
	8		∴ if G > = or > L, K > = or < N.
	9	C. 1.	And G, K, are equims. of A, F, and L, N, of B, F ;
	10	Def. 5, V.	∴ A : B = E : F.
	11	Rec.	∴ *Ratios are the same, &c.* Q. E. D

Otherwise. *Hyp.*. If A : B = C : D, and C : D = E : F; then A : B = E : F; or, if 3 : 6 = 1 : 2; and 1 : 2 = 4 : 8, then 3 : 6 = 4 : 8.

Alg. Of antecedents take mA, mC, mE;
and of consequents „ n B, n D, n F.
∵ A : B = C : D; if m A $>$ = or $<$ n B,
∴ m C $>$ = or $<$ n D.
Again, ∵ C : D = E : F; if m C $>$ = or $<$ n D,
∴ m E $>$ = or $<$ n F;
Now m A, m E equims. of A, E; n B, n F, of B, F;
∴ by Def. 5, V A : B = E : F.

Arith. Take the *ms* 2×3, 2×1, 2×4;
and 2×6, 2×2, 2×8;
∵ 3 : 6 = 1 : 2, if $2\times3 < 2\times6$,
∴ $2\times1 < 2\times2$.
Again, ∵ 1 : 2 = 4 : 8, if $2\times1 < 2\times2$,
∴ $2\times4 < 2\times8$.
Now 2×3, 2×4 are equims; also 2×6 and 2×8;
∴ 3 : 6 = 4 : 8.

SCH. This Proposition is to Ratios. what Prop. 30, Bk. 1, is to Parallel lines; Ax. 1, Bk. I, to Magnitudes; and Ax. 1, Bk. V. to Equimultiples.

COR. 1. If A : B = C : D, but C : D $>$ or $<$ E : F, then A : B $>$ or $<$ E : F.

For, whatever part of D be contained in C, a greater or less number of times than the like part of F is contained in E, the like part of B must be contained in A the same greater or less number of times.

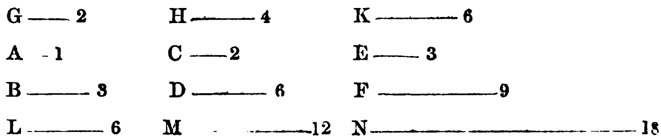
COR. 2 Thus also, if A : B $>$ or $<$ C : D, and C : D = E : F, then A : B $>$ or $<$ E : F.

PROP. XII.—THEOR.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

CON. Pst. 1, V. DEM. Def. 5, V. Definition of Proportion. 1, V. If any no. of *Ms* be equims of as many, each of each, what m soever any one of them is of its *pt.*, the same m shall all the first *Ms* be of all the others.

E.	1	Hyp.	Let A : B = C : D = E : F;
	2	Conc.	then A : B = A+C+E : B+D+F.



C.	1	Pst. 1, V.	Of A,C,E, take equims. of G,H,K;
	2	,,	and of B,D,F, equims. L,M,N.
D.	1	Hyp.	then ∵ A : B = C : D = E : F,
	2	C. 1.	and G,H,K, are equims of A,C,E,
	3	C. 2.	and L,M,N, equims. of B,D,F;
	4	Def. 5, V.	∴ G > = or < L, H also > = or < M, and K > = or > N;
	5		wherefore if G > = or < L,
	6		then G + H + K > = or < L + M + N.
	7	C. 1.	But G, and G + H + K, are equims. of A, and A + C + E.
	8	1, V.	for whatever *m* of A, G is, the same *m* are all, G + H + K of A + C + E;
	9	D. 7 & 8.	*i. e.* G, and G + H + K are equims. of A, and A + C + E.
	10	*Sim.*	So L, and L + M + N are equims. of B, and B + D + F.
	11	Def. 5, V.	∴ A : B = A + C + E : B + D + F.
	12	Rec.	Wherefore, *if any number of magnitudes, &c.* Q. E. D.

Otherwise. *Hyp*. A : B = C : D = E : F; then A : B = A + C + E : B + D + F; or 1 : 3 = 2 : 6 = 3 : 9, then 1 : 3 = 1 + 2 + 3 : 3 + 6 + 9, or 1 : 3 = 6 : 18.

Alg. I. Of the Antecs. take mA, mC, mE,
Of the Conseqs. ,, nB, nD, nF;
∵ A : B = C : D, ∴ if m A > = or < n B, m C, > = or < n D;
∵ C : D = E : F, ∴ if m C > = or < n D, mE > = or < nF:
∴ If m A > = or < n B, m A + m C + mE > = or < n B + n D + n F.
Now (1, V.) m A + mC + mE = m (A + C + E),
so that m A, m A + m C + m E are equims. of A, and A + C + E;
and nB, nB + nD, + n F are equims. of B, and B + D + F;
∴ A : B = A + C + E : C + D : F.

Arith. Of Antecs. take $3 \times 1, 3 \times 2, 3 \times 3,$

Of Conseqs. „ $2 \times 3, 2 \times 6, 2 \times 9;$

$\because 1 : 3 = 2 : 6, \therefore$ if $3 \times 1 > = $ or $< 2 \times 3$, $3 \times 2 > =$ or $< 2 \times 6$;

and $\therefore 2 : 6 = 3 : 9, \therefore$ if $3 \times 2 > = $ or $> 2 \times 6, 3 \times 3 > =$ or $> 2 \times 9$.

Now $(3 \times 1) + (3 \times 2) + (3 \times 3) = 3\,(1 + 2 + 3)$,

so that $3 \times 1, (3 \times 1) + (3 \times 2) + (3 \times 3)$ are equims. of 1 and $(1 + 2 + 3)$

and 2×3, $(2 \times 3) + (2 \times 6) + (2 \times 9)$ equims. of 3 and $(3 + 6 + 9)$;

$\therefore 1 : 3 = (1 + 2 + 3) : (3 + 6 + 9)$, or $1 : 3 = 6 : 18$.

Alg. II. For Antecs. take a, and $a, b, c, d, e,$ &c.

For Conseqs. „ a', and a', b', c', d', e', &c.

By given Hyp. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'} =$ &c.

Take product of extremes and means; $ab' = a'b$, $ac' = a'c$, $ad' = a'd$, $ae' = a'e$, &c.

Add Antecs. for numerator, Conseqs. for denominator,

$$\frac{a + b + c + d + e + \&c.}{a' + b' + c' + d' + e' + \&c.};$$

Divide by $\frac{a}{a'}$, and we have, $\frac{aa' + a'b + a'c + a'd + a'e\ \&c.}{aa' + ab' + ac' + a'd + a'e\ \&c.}$

Now $\frac{a}{a} = 1$, & $\frac{aa' + a'b + a'c + a'd + a'e\ \&c.}{aa' + ab' + ac' + ad' + ae'\ \&c.} = 1$.

And the *quotients* in each case being *unity*,

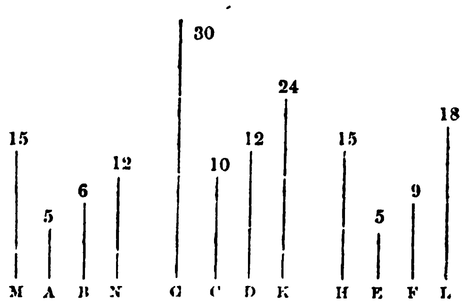
$\therefore a : a' = a + b + c + d + e\ \&c. : a' + b' + c' + d' + e'\ \&c.$

Prop. XIII.—Theor.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Con. Pst. 1, V. Dem. Def. 7, V. Definition of greater and less ratio Def. 5, V. Definition of Proportion.

E.	1	Hyp.	Let A : B = C : D, but C : D > E : F;
	2	Conc.	then A : B > E : F.



C.	1	H.	∵ C : D > E : F,
	2	Def. 7, V.	∴ some equims of C the 1st and E the third, and some of D the 2nd and F the 4th; so that *m* of C > *m* of D. but *m* of E ≯ *m* of F;
	3	Pst. 1, V.	Take such equims. G & H of C & E, K & L of D & F, so that G > K, but H ≯ L;
	4	Pst. 1, V.	Whatever *m*, G is of C, take M of A; and whatever *m*, K is of C, take N of B.
D.	1	H&C. 3, 4.	∵ A : B = C : D, and ∵ M, G, equims. of of A, C; N, K of B, D;
	2	Def. 5, V.	∴ if M > = or < N, G > = or < K.
	3	C. 3.	but G > K ∴ M > N.
	4	C. 3 & 4.	But H ≯ L, and M, H equims of A, E;— N, L of B, F;
	5	Def, 7, V.	∴ A : B > E : F.
	6	Rec.	Wherefore *if the first has to the second &c.* Q. E. D.

Cor. If A : B > or < B : E, but C : D = E : F, it may also be demonstrated that A : B > or < E F.

Alg. & Arith Hyp. Let A 5 : B 6 = C 10 : D 12; but C 10 : D 12 > E 5 : F 9; then A 5 : B 6 > E 5 : F 9. Let $m = 3$, & $n = 2$.

Alg. ∵ C : D > E : F,
Assume m C, n D, m E, n F, so that
if m C > n D, m E ≮ n F.

Now $\because$ A : B = C : D,—if m C > n D, m A > nB;
$\therefore$ m A > n B, and m E < n F,
$\therefore$ A : B > E : F.

Arith. $\because$ 10 : 12 > 5 : 9,
Assume 3 × 10 > 2 × 12, but 3 × 5 < 2 × 9.
Now $\because$ 5 : 6 = 10 : 12, if 3 × 10 > 2×12; and 3 × 5 > 2× 6
$\therefore$ 3 × 5 > 2 × 6, and 3 × 5 < 2 × 9;
$\therefore$ 5 : 6 > 5 : 9.

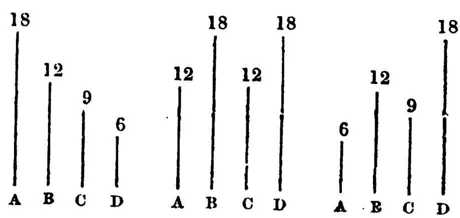
SCH. "This proposition is equivalent to stating; 1°. that if any ratio be greater than another, every ratio which is equal to the former will also be greater than the latter; 2°. Also, that if one ratio be greater than another, every ratio which is greater than the former is also greater than the latter." LARDNER.

PROP. XIV.—THEOR.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

DEM. 8, V. The gr. M. a gr. ratio. 13, V.—10, V. one M with a gr. ratio to a third. 9, V. Two Ms. each with the same ratio to a third.

E. 1	Hyp.	Let A : B = C : D,
2	Conc.	If A > = or < C, B > = or < D.



CASE I. Let A > C, B will be > D.

D. 1	Hyp.	$\because$ A : B = C : D, or C : D = A : B;
2	8, V.	and if A > C, A : B > C : B;
3	13, V.	$\therefore$ C : B > C : B.
4	10, V.	But the *M* to which the same has the gr. ratio is the less,
5	Conc.	$\therefore$ D < B, *i. e.*, B > D.

CASE II. If A = C, B = D.

D. 1	Hyp. & 9,V.	$\because$ A : B = C, *i. e.* A : D, $\therefore$ B = D.

CASE III. If A < C, B < D.

D.1	Hyp.	$\because$ C > A, and C : D = A : B;
2	Case 1.	$\therefore$ D > B, *i. e.* B < D.
3	Rec.	Therefore, *if the first has the same ratio,&c.* Q.E.D.

Briefly. 1°. Let A > C; then, by 8, V., A : B > C : B; but A : B = C : D, $\therefore$ by 13, V., C : D > C : B, and $\therefore$ by 10, V., B > D.

Similarly, 2°. if A = C, by 9, V., B = D, and 3°. if A < C, B < D.

COR. Hence also, if A : B = C : D, and if the 2nd B > = or < the 4th D, then the 1st A > = or < the 3rd C.

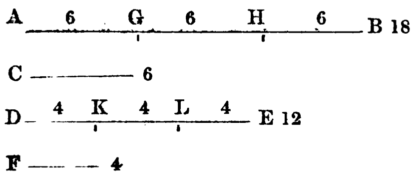
PROP. XV.—THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

CON. Pst. 2, V. Any given *m* of a M may be divided into parts, each of which is equal to that M.

DEM. 7, V. Equal Ms, with the same ratio. 12, V. Antecedents to consequents.

E. 1	Hyp.	Let A B be the same m of C, that D E is of F;
2	Conc.	then C : F = A B : D E.



C. 1	Pst. 2, V.	Divide A B into Ms each = C, *i. e.*, A G = G H = H B = C.
2	" "	& " D E " each = F, *i. e.*, D K = K L = L E = F.
D. 1	C.	Now A G, G H, H B in *number* = D K, K L, & L E;
2	C. 1 & 2.	and A G = G H = H B, and D K = K L = L E,
3	7, V.	∴ A G : D K = G H : K L = H B : L E.
4	12, V.	And ∵ A G : D K = A G + G H + H B : D K + K L + L E,
5	C. 1 & 2.	but A G = C, and D K = F;
6	Conc.	∴ C : F = A B : D E.
7	Rec.	Wherefore, *Magnitudes have the same ratio*, &c. Q. E. D.

Otherwise, Take m any number, A & B two magnitudes; then A : B = m A : m B.

D. 1	7, V. & 12, V.	∵ A : B = A : B, ∴ A : B = A + A : B + B, or 2 A = 2 B;
2	D. 1,	and ∵ A : B = 2 A : 2 B,
3	12, V.	∴ A : B = 2 A + A : 2 B + B, or 3 A = 3 B.
4	*Sim.*	And so on for all equims. of A and B;
5	Conc.	∴ A : B = m A : m B.

Cor. 1. *Magnitudes have the same ratio to one another which their equal submultiples or like parts have.*

thus, $A : B = \frac{A}{3} : \frac{B}{3}$; or, $12 : 18 = \frac{12}{3} : \frac{18}{3}$, *i. e.*, 12 : 18 = 4 : 6.

Cor. 2. In Proportionals the equims. of the 1st and 2nd have the same ratio as the equims. of the 3rd and 4th. If A : B = C : D, then m A : m B = n C : n D;

Arith. If $2 : 3 = 4 : 6$, then $2 \times 2 : 2 \times 3 = 3 \times 4 : 3 \times 6$;
i. e., $4 : 6 = 12 : 18$.

Cor. 3. In Proportionals also, any like parts of the 1st and 2nd, and also of the 3rd and 4th are proportional; as

$$\frac{A}{2} : \frac{B}{2} = \frac{C}{3} : \frac{D}{3}.$$

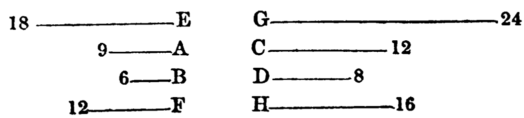
Sch. The Proportion is sometimes explained, "one magnitude shall have the same ratio to another magnitude of the same kind which any multiple of the first has to the same multiple of the second."—Hose.

Prop. XVI.—Theor.

Alternando, *or* Permutando. *If four Magnitudes* of the same kind *be proportionals, they shall also be proportionals when taken alternately.*

Con. Pst. 1, V. Dem. 15, V.—11, V.—14, V.

E. 1	Hyp.	Let $A : B = C : D$;
2	Conc.	then $A : C = B : D$.



C.	Pst. 1, V.	Take E, F, equims. of A, B, and G, H, equims. of C, D.
D.1	C.	∵ E same *m* of A, that F is of B,
2	15, V.	∴ $A : B = E : F$;
3	H. & 11, V.	but ∵ $A : B = C : D$, ∴ $C : D = E : F$.
4	C.	Again ∵ G & H are equims. of C & D,
5	15, V.	∵ $C : D = G : H$.
6	D.3, & 11, V	but $C : D = E : F$, ∴ $E : F = G : H$.
7	14, V.	In 4 proportionals, if 1st, $>$ $=$ or $<$ 3rd, the 2nd is $>$ $=$ or $<$ 4th;

8		∵ if E $>=$ or $<$ G, F $>=$ or $<$ H;
9	C.	Now E, F, are equims. of A, B; & G, H, of C, D;
10	Def. 5, V.	∴ A : C = B : D.
11	Rec.	*If* then *four magnitudes of the same kind, &c.*

Q. E. D.

Or,

C.	Pst. 1, V.	Take m A, m B, n C, n D, equims. of A, B, C, D.
D. 1	15, V. & 11, V,	∵ m A : m B = A : B, ∴ m A : m B = C : D.
2	15, V. & 11, V.	also, ∵ n C : n D = C : D, ∴ m A : m B = nC : nD;
3	14, V.	now m B $>=$ or $<$ nD, as m A $>=$ or $<$ nC;
4	Def. 5, V.	∴ A : C = B : D.

Alg. & Arith. Hyp. Let $\frac{a}{b}, \frac{2}{4}, = \frac{c}{d}, \frac{3}{6}$; then $\frac{a}{c}, \frac{2}{3} = \frac{b}{d}, \frac{4}{6}$.

Alg. Let $\frac{a}{b} = \frac{c}{d}$, then $a\,d = b\,c$.

($\div$ by $c\,d$), and $\frac{ad}{cd} = \frac{bc}{cd}$, or, $\frac{a}{c} = \frac{b}{d}$.

Arith. If $\frac{2}{4} = \frac{3}{6}$, then $2 \times 6 = 4 \times 3$.

($\div$ by 3×6,) and $\frac{2 \times 6}{3 \times 6} = \frac{4 \times 3}{3 \times 6}$, or $\frac{2}{4} = \frac{4}{6}$.

Use & App I. From principles established, especially from Def. 7, V,—and from Props. 7, 11, 14, & 15, bk. V, the following Theorem, *Chambers' Ex. p.* 56, may be demonstrated;

If to the terms of a ratio, A : B, *the same magnitude,* C, *be added, the ratio will be* unchanged, increased, or diminished, *according as it is a ratio* of equality, *of* less inequality, *or of* greater inequality.

Case I. If A = B, then A + C : B + C = A : B.

D. 1	Ax. 2, I.	∵ A = B, ∴ A + C = B + C;
2	7, V. & 16, V.	∴ A + C : C = B + C : C, and A + C : B + C = C : C.
3	*Sim.* & 11, V.	So A : B = C : C. and ∴ A + C : B + C = A : B.

Case II. If A $<$ B, then A + C : B + C $>$ A : B.

D. 1.	H.	∵ A $<$ B, if B − A = D; then B = A + D.
2	*Sup.*	Let p C $>$ A, and n D the least m of D exceeding p A, so that p A $<$ n D;
3	D. 2.	hence p A + A $>$ n D, or = n D;
4	D. 2, 3.	& ∴ p A + p C $>$ n D, or p (A + C) $>$ n D.
5	*Sup.*	Next, let $m = n + p$, or $p = m - n$;
6	D. 2, 5, & 1.	then ∵ p A $<$ n D; $p = m - n$, and D = B − A :
7		∴ $(m-n)$ A $<$ n (B−A); or m A − n A $<$ n B − n A:

8	*Add.*	To both sides add n A, ∴ m A $<$ n B.
9	D. 4.	Again. ∵ p (A + C) $<$ n D,
10	D. 6.	∴ p A + p C $>$ n (B−A); or m A−n A + m C−n C $>$ n B−n A.
11	*Add.*	Add n A + n C to each side of the equation;
12	Ax. 4, I.	∴ m A + m C $>$ n B + n C; or m (A + C) $>$ n (B + C);
13	D. 8. Def. 7, V.	but m A $<$ n B, ∴ A + C : B + C $>$ A : B.

CASE III. If A $>$ B, then A + C : B + C $<$ A : B.

D. 1.	H.	∵ A $>$ B, if A−B = D, then A = B + D.
2	*Sup.*	Let p C $>$ B; and m D the least m of D, so that p B $<$ m D;
3	D. 2.	hence p B + B $>$ or = m D;
4	D. 2 & 3,	& ∴ p B + p C $>$ m D, or p (B + C) $>$ m D.
5	*Sup.*	Next, let n = m + p; or p = n−m;
6	D. 2, 5, & 1.	then ∵ p B $<$ m D, p = n−m, and D = A−B,
7		∴ (n−m) B $<$ m (A−B); or n B−m B $<$ mA−m B;
8	*Add.*	Add m B, ∴ n B $<$ m A; m A $>$ n B.
9	D. 4.	Again ∵ p (B + C) $>$ m D,
10	D. 6.	∴ p B + p C $>$ m (A−B); or n B−m B + n C−m C $>$ m A−m B;
11	*Add.*	Add m B + m C to each side of the equation;
12	Ax. 4, I.	∴ n (B + C) $>$ m (A + C); or m (A + C) $<$ n (B + C).
13	D. 8, Def. 7, V.	But m A $>$ n B, ∴ A + C : B + C $<$ A : B.
14	Rec.	Wherefore, *If to the terms of a ratio &c.*

II. Props. 11, 15 & 16, bk. V., also furnish the principles by which to establish the useful Theorem,—*Chambers' Ex. p.* 54;—that,

"*If all the terms, or any two homologous terms, or the terms of either of the ratios of a proportion, be multiplied or divided by the same number, the resulting magnitudes will remain proportional.*

E. 1	Hyp.	Let A : B = C : D, and m, n be any two numbers; then,

CASE I. m A : m B = m C : m D;

D. 1	15, V.	∵ A : B = m A : m B; and C : D = mC : m D;
2	11, V.	∴ m A : m B = m C : m D.

CASE II. $\frac{1}{m}$A : $\frac{1}{m}$B = $\frac{1}{m}$C : $\frac{1}{m}$D.

D. 1	H.	∵ A & B are mults. of $\frac{1}{m}$A $\times$ m, and of $\frac{1}{m}$B $\times$ m;
2	15, V.	∴ A : B = $\frac{1}{m}$A : $\frac{1}{m}$B.
3	*Sim.*	In like manner C : D = $\frac{1}{m}$C : $\frac{1}{m}$D.
4	11, V.	∴ $\frac{1}{m}$A : $\frac{1}{m}$B = $\frac{1}{m}$C : $\frac{1}{m}$D.

Case III. $mA : B = mC : D$.

D. 1	H. & Sch. 3, V.	$\because A : B = C : D$, $\therefore mA : B = mC : D$;
2	*Sim.*	In like manner, $A : nB = C : nD$.

Case IV. $\frac{1}{m}A : B = \frac{1}{m}C : D$.

D. 1	H. & 16, V.	$\because A : B = C : D$, and $A : C = B : D$;
2	Case 2,	$\therefore \frac{1}{m}A : \frac{1}{m}C = B : D$,
3	16, V.	and $\therefore \frac{1}{m}A : B = \frac{1}{m}C : D$.

Case V. $mA : mB = C : D$.

D. 1	H. & 15, V.	$\because A : B = C : D$, and $A : B = mA : mB$;
2	11, V.	$\therefore mA : mB = C : D$.

Case VI. $\frac{1}{m}A : \frac{1}{m} = C : D$.

D. 1	H.	$\because A : B = C : D$, & A, B are equims. of $\frac{1}{m}A$, $\frac{1}{m}B$ by m;
2	15, V.	$\therefore A : B = \frac{1}{m}A : \frac{1}{m}B$.
3	11, V.	and $\therefore \frac{1}{m}A : \frac{1}{m}B = C : D$.
4	Rec.	Wherefore, *If all the terms, &c.* Q. E. D.

Prop. XVII,—Theor.

Dividendo. *If magnitudes taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them, the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.*

N.B. The General Enunciation of this 17th Proposition is variously given;

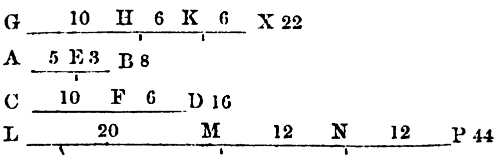
"If magnitudes be proportional, they will also be proportional *by division.*"—Euclid.

"If four magnitudes, A, B, C, D, be proportionals, they shall also be proportionals, when taken *dividedly*: that is, the difference of the first and second shall be to the second as the difference of the third and fourth to the fourth; or *dividendo*, A ∾ B : B = C ∾ D : D."—De Morgan.

"If four magnitudes be proportional, the first being greater than the second and the third greater than the fourth; then the excess of the first above the second shall be to the second, as the excess of the third above the fourth is to the fourth."—Hose.

Con. Pst. 1, V. Dem. 1, V.—If any number of *Ms* be equims. of as many, &c., 2, V. If the 1st M be the same *m* of the 2nd, &c. Def. 5, V. Criterion of the equality of two ratios. Ax. 4, I. If equals be added to unequals the wholes are unequal. Ax. 5, I. If equals be taken from unequals the remainders are unequal.

E 1	Hyp. 1	Let A B, B E, C D, D F, be *Ms* taken jointly and proportionals,
		i. e., A B : B E = C D : D F;—A B > B E and C D > D F;
2	„ 2	and let A E be the excess of A B above B E, and C F the excess of C D above D F;
3	Conc.	then A E : E B = C F : F D.



C. 1	Pst. 1, V.	Of A E, E B, C F, F D take equims. G H, H K, L M, M N;
2	„ „	and of E B, F D any equims. X K, N P.
D. 1	C.	∵ G H, H K are equims. of A E, B E;
2	1, V.	∴ G H same *m* of A E that G K is of A B;
3	C. 1.	But G H same *m* of A E that L M is of C F;
4		∴ G K „ A B „ L M „ C F.
5	C. 1.	Again ∵ L M same *m* of C F that M N is of F D;
6	1, V.	∴ L M „ of C F „ L N „ C D;
7	D. 4.	but L M same *m* of C F that G K is of A B,
8	D. 6 & 7.	∴ G K, L N are equims of A B, C D.
9	C. 1.	Next, ∵ H K, M N are equims of E B, F D;
10	C. 2.	and ∵ K X, N P equims, of E B, F D,
11	2 V.	∴ H K + K X & M N + N P equims. of E B, F D;
		i. e. H X and M P are equims of E B, F D;

12	Hyp.	And $\because$ AB : BE = CD : FD;
13	D. 8.	and $\because$ GK, LN are equims. of AB, CD,
14	D. 9.	and HX, MP equims. of BE, FD;
15	Def. 5, V.	$\therefore$ if GK $>=$ or $<$ HX, then LN $>=$ or $<$ MP;
16	*Sup. Add.* Ax., 5 I.	but if GH $>$ KX, add HK to both, $\therefore$ GK $>$ HX;
17		$\therefore$ also LN $>$ MP;
18	*Sub.*	take MN from both; and LM $>$ NP;
19	Def. 5, V.	$\therefore$ if GH $>=$ or $<$ KH, LM $>=$ or $<$ NP.
20	C. 1.	But GH, LM are equims. of AE, CF,
21	C. 2.	and KX, NP are equims. of EB, FD;
22	Def. 5, V.	$\therefore$ AE : EB = CF : FD.
23	Rec.	Wherefore, *If Magnitudes taken jointly, &c.*

Q. E. D.

Otherwise. If A + B : B = C + D : D, then *by division* A , B = C : D;

C. 1	Pst. 1, V.	Take mA, nB, mults. of A, B.
D. 1	*Sup.* 1.	Let m A, $>$ nB.
2	*Add.* Ax. 4, I,	To both sides add mB; $\therefore$ mA + m B $>$ mB + nB, or $(m+n)$ B,
3	II.	but $\because$ A + B : B = C + D : D,
4		$\therefore$ if m (A + B) $>$ $(m+n)$B, m (C + D) $>$ $(m+n)$ D;
5		thus mC + m D $>$ mD + n D;
6	*Sub.* Ax. 5, I.	from both sides take m D, $\therefore$ m C $>$ nD; *i. e.*, if m A $>$ n B, m C $>$ n D.
7	*Sup.* 2. *Sim.*	If m A = n B, then m C = n D,
8	*Sup.* 3. *Sim.*	and if m A $<$ n B, then m C $<$ nD.
9	Def. 5, V.	$\therefore$ A : B = C : D, by division Def. 16, V.
10	Rec.	Wherefore, *if magnitudes, &c.*

Alg. and Arith. Hyp. Take a, 8 : b, 2 : : c, 12 : d, 3.

Alg. Here $\frac{a}{b} = \frac{c}{d}$,

Sub 1 from each; $\frac{a}{b} - 1 = \frac{c}{d} - 1$,

i. e. $\frac{a-b}{b} = \frac{c-d}{d}$,

or $a-b : b = c-d : d$.

Arith. $\frac{8}{2} = \frac{12}{3}$,

Sub. 1 $\frac{8}{2} - 1 = \frac{12}{3} - 1$,

i. e. $\frac{8-2}{2} = \frac{12-3}{3}$;

or $8-2 : 2 = 12-3 : 3$, or $6 : 2 = 9 : 3$.

COR. 1. Convertendo.—*If four Ms*, A,B,C,D be *proportionals* $A : B = C : D$, *they shall also be proportionals by conversion, i.e.*, the 1st : 1st∼2nd = 3rd : 3rd∼4th,—or $A : A \sim B = C : C \sim D$.

For, *invertendo*, Pr. B,V. $B : A = D : C$;
dividendo, 17, V. $B \sim A : A = D \sim C : C$;
invertendo, B, V. $A : A \sim B = C : C \sim D$; *or convertendo.*

COR. 2. *If four Ms of the same kind*, A,B,C,D, *be proportionals*, $A : B = C : D$, *then the greatest + the least are together greater than the other two.*

D. 1	Sup. 1.	Let A one of the extremes be the greatest; as $12 : 8 = 3 : 2$.
2	14, V.	$\because B < A, \therefore D < C$; and $\because C < A, \therefore D < B$ $\therefore$ D is the least.
3	H. & 17, V.	$\therefore A : B = C : C, \therefore A - B : B = C - D : D$.
4	D. 2,	and $\because B > D, \therefore A - B > C - D$.
5	Add.	To each add $B + D$; $\therefore A + D > C + B$; *i. e.* the greatest 12 + the least 2 > the other two 8 + 3.
6	*Sup.* 2.	Let B one of the means be greatest; as $4 : 16 = 3 : 12$;
7	Pr. B, V.	*invert.* $B : A = D : C$.
8	*Sim.*	So, as before, $B + C > A + D$, *i. e.* the greatest B 16 + the least C 3 > the other two 4 + 12.
9	Rec.	$\therefore$ the sum of the greatest and least > the sum of the other two.

N. B.—This Corollary is identical with Prop. 25, Bk. V.

COR. 3. *In three Proportionals*, $A : B : C$, as $2 : 4 : 8$, or $9 : 6, 4$, *the sum of the extremes*, $A + C$ *is greater than twice the mean*, 2 B; *and therefore half the sum*, $\frac{A + C}{2}$ *of the extremes is greater than the mean*, B.

For, if the mean $B > C$, then $B < A$; and if $B < C$, then $B > A$.
$\therefore$ the extremes A & C are the greatest M, and the least.
and as before, in Cor. 2, $A + C > B + B$, or 2 B;

$$\text{and } \therefore \frac{A + C}{2} > B.$$

Use & App. When half the sum of two Ms, $\frac{A + C}{2}$, is as much greater than the one, as it is less than the other, ***that half sum*** is an ***arithmetical mean*** between the two; thus, in $\frac{2+8}{2} = 5$, $8-5 = 3$, & $5-2=3$; and $2 : 5 : 8$ are in arithmetical progression. Except when the magnitudes are equal, the arithmetical mean between two magnitudes, A and C, is therefore greater than the geometrical mean, *i. e*, the arith. mean, $\frac{A + C}{2} > B$ the geom. mean; or $\frac{8+2}{2}$ *i. e.*, $5 > 4$. *Geom. Plane, Sol. & Sph. pp* 41 & 42.

Prop. XVIII.—Theor.

Componendo. *If magnitudes taken separately, be proportionals, they shall also be proportionals when taken jointly,* by composition; *that is, if the first be to the second, as the third is to the fourth, the first and second together shall be to the second, as the third and fourth together to the fourth.*

"If magnitudes be proportional, they will also be proportional *by composition.*"—Euclid.

"The terms of an analogy are proportional *by composition.*"—Bell.

Con. Pst. 1, V. Dem. Def. 5, V. Criterion of the equality of ratios.

Ax. 3, V. A *m* of a gr. M is gr. than the same *m* of a less.

5, V. If one M be the same *m* of another which a M taken from the 1st is of a M taken from the other; the rem. is the same *m* of the rem. that the whole is of the whole.

6, V. If two Ms be equims. of two others, and if equims. of these be taken from the first two; the rems. either = these others, or are equims.

Cor. 4, V. If 1st : 2nd = 3rd : 4th, then any equims. of 1st and 3rd the same ratio to 2nd and 4th; and 1st : 3rd = equim. 2nd : equim 4th.

Pr. A, V. If 1st : 2nd = 3rd : 4th; then if 1st > 2nd, 3rd > 4th, and if =, =; if <, <.

Ax. 2, V. Those Ms of which the same or equal Ms are equims. are equal.

Ax. 4, V. That M of which a *m* is greater than the same *m* of another is gr. than that other M.

E.	1	Hyp.	Let AE : EB = CF : FD.
	2	Conc.	then AE + EB = : BE = CF + FD : DF; *i. e.*, AB : BE = CD : DF.
C.	1	Pst. 1, V.	Of AB, BE, CD, DF take any equims. GH, HK, LM, MN;
	2	”	and of BE, DF any equims. KO, NP.

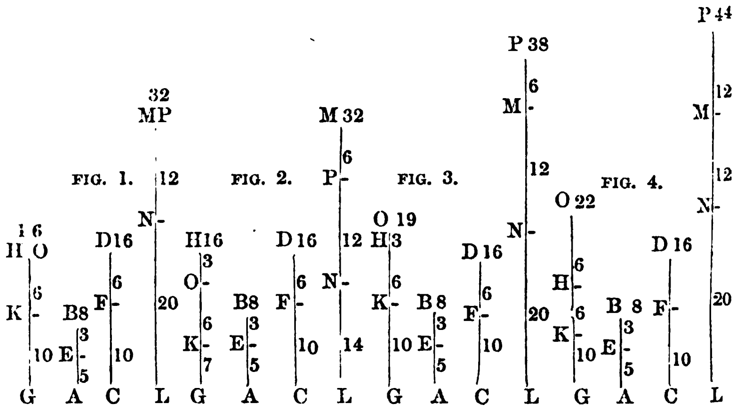


FIG. 1. FIG. 2. FIG. 3. FIG. 4.

D.	1	C. 2.	∵ KO, NP are equims of BE, DF, & KH, NM also equims of BE. DF;
	2	Def.. 5, V.	∴ if KO > = or < KH, then NP > = or > NM.

Case I. Let KO ≯ KH, & ∴ NP ≯ NM. *Fig.* 1 & 2.

D.	1	C. 1.	∵ GH, HK equims of AB, BE, AB being gr. than BE;
	2	Ax. 3, V.	∴ GH > HK,
	3	H. Ax. 3, V.	but KO ≯ KH, ∴ GH > KO.
	4	*Sim.*	In the same manner LM > NP;
	5	H.	∵ KO ≯ KH, ∴ GH, a *m* of AB always > KO the same *m* of BE;
	6		∴ also LM the *m* of CD > NP the *m* of DF.

Case II. Let KO $>$ KH, and $\therefore$ NP $>$ NM. *Fig.* 3.

D. 1	C. 1.	And $\because$ the whole GH is the same *m* of the whole AB as HK of BE,
2	5, V.	$\therefore$ rem. GK same *m* of rem. AE as GH of AB, or as LM of CD.
3	*Sim.*	$\because$ LM the same *m* of CD, as MN of DF,
4	5, V.	$\therefore$ rem. LN same *m* of rem. CF as the whole LM of the whole CD.
5	D. 2.	But LM same *m* of CD, as GK of AE;
6	6, V.	$\therefore$ GK same *m* of AE, as LN of CF, *i. e.* GK and LN are equims. of AE & CF.
7	C. 2.	And $\because$ KO, NP are equims. of BE, DF;
8	Pst. 1, V.	& $\because$ from KO, NP may be taken KH, NM also equims. of BE, DF,
9	6, 5.	$\therefore$ rems. HO, MP either $=$ rems. BE, DF, or are equims. of BE, DF.

Subdivision 1 Let HO, MP $=$ BE, DF. *Fig.* 3.

D. 1	H & Case 2, D. 6.	$\because$ AE : EB $=$ CF : FD, & GK, LN are equims. of AE, CF;
2	Cor. 4, V.	$\because$ GK : EB $=$ LN : FD;
3	H.	But HO$=$ EB & MP $=$ DF, $\therefore$ GK : HO $=$ LN : MP;
4	Pr. A, V.	$\therefore$ if GK $>$ $=$ or $<$ HO, LN $>$ $=$ or $<$ MP.

Subd. 2. Let HO, MP be equims. of EB, FD. *Fig.* 4.

D. 1	H. & Case 2, D. 6.	$\because$ AE : EB $=$ CF : FD, and of AE, CF, the equims are GK, LN,
2	H.	& of EB, FD the equims are HO, MP;
3	Def. 5, V.	$\therefore$ if GK $>$ $=$ or $<$ HO, LN $>$ $=$ or $>$ MP;
4	*Sub.* & Ax. 5, I.	but if GH $>$ KO, from each taking KH, then GK $>$ HO;

5		$\therefore$ also LN > MP;
6	*Add.* Ax. 4, I.	to both add NM $\therefore$ LM > NP;
7		$\therefore$ if GH > KO, LM > NP.
8	*Sim.*	So, if GH = or < KO, LM = or < NP;
9	Case I. D. 3, 4.	And when KO $\ngtr$ KH, then GH > KO & LM > NP;
10	C.	but GH, LM equims of AB, CD, & KO, NP of BE, DE;
11	Def. 5, V.	$\therefore$ AB : BE = CD : DF, *i. e.*, AE + EB : BE=CF + FD : DF,
12	Rec.	Wherefore, *if magnitudes taken separately &c.* Q. E. D.

Otherwise. If A : B = C : D ;— and of B, D any like pts, as the *nth*, be contained in A. C, *m* times exactly, or with like remainders ; *i. e.* if *m* B = A, & *m* D = C,— then those parts will be contained in A + B & C + D, the same number of times exactly, *m* + *n* with the same remainders ;
$\therefore$ by Def. 5,V. A + B : B = C + D : D.

Cor, A : A + B = C : C + D.

By *indirect* Proof.

E.	1	Hyp.	If A : B = C : D,
	2	Conc.	then A + B : B=C + D : D.
D.	1	*Sup.*	If C + D : D $\neq$ A + B : B,
	2		let $c + d : d$ = A + B : B, and $d \neq$ D.
	3	H. 17, V.	$\because$ A + B : B = C + d : d, $\therefore$ A : B = C : d.
	4	H. 11, V.	but A : B = C : D, $\therefore$ C : D = C : d.
	5	9, V.	$\therefore$ D = d, contrary to the hypothesis;
	6	Conc.	$\therefore$ A + B : B = C + D : D.

Alg. & Arith. Hyp. $a\ 5 : b\ 3 = c\ 10 : d\ 6.$

Alg. Let $a : b = c : d,$

$$(+1) \qquad \frac{a}{b} + 1 = \frac{c}{d} + 1.$$

$$i.\ e.\ \frac{a+b}{b} = \frac{c+d}{d},$$

or, $a + b : b = c + d : d.$

Arith. Let $5 : 3 = 10 : 6,$

$$(+1) \qquad \frac{5}{3} + 1 = \frac{10}{6} + 1,$$

$$i.\ e.\ \frac{5+3}{3} = \frac{10+6}{6};$$

or, $8 : 3 = 16 : 6.$

Use & Appl. This method of reasoning is often employed. By an extension of it, as indicated in the Corollary, we find that the terms of a proportion are also proportional by addition. For, Let A : B = C : D,—then, *addendo*, A : A + B = C : C + D;

Invertendo. B : A=D : C; *Componendo*, 18, V. A+B : A = C+D : C; ∴ *invert.* (A, V.) A : A + B = C : C + D.

Prop. XIX.—Theor.

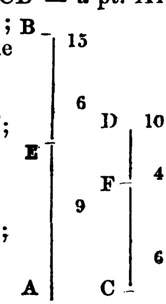
If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall be to the remainder as the whole to the whole.

"If a whole be to a whole as a part taken away to a part taken away; the part left is to the part left, as the whole is to the whole." Euclid.

"If four magnitudes, which are all of the same kind, be proportional, the first being greater than the third, and the second than the fourth; then the excess of the first above the third, shall be to that of the second above the fourth as the second is to the first."—Hose.

Dem. 16, V. *Alternando*;— 17; V. *Dividendo*; 11, V. Ratios the same to the same ratio are the same to one another.

E. 1	Hyp.	Let the whole AB : the whole CD = a pt. AE from AB : CE a pt. from CD;
2	Conc.	then rem. EB : rem. FD = the whole AB : the whole CD.
D. 1	H. & 16, V.	∵ AB : CD = AE : CF, ∴ *alt.* BA : AE=DC : CF;
2	17, V.	& ∵ Ms jointly are propl.,— separately they are also;
3	16, V.	∴ BE : EA = DF : FC, & *alt.* BE : DF = EA :FC;
4	H.	but A E : CF = AB : CD;
5	11, V.	∴ rem. BE : rem. DF = integer AB : integer CD.
6	Rec	∴ *If a whole magnitude, &c.* Q. E. D.



Otherwise,

D. 1	Hyp.	$\because$ A : B = C : D, C being $<$ A & D $<$ B ;
2	16, V. & 17, V.	$\therefore$ *alt.* A : C = B : D, & *div.* A − C : C = B − D : D.
3	16, V.	Again *alt.* A − C : B − D = C : D.
4	Conc.	but A : B = C : D, $\therefore$ A − C : B − D = A : B.

Cor. 1. *Also the remainder shall be to the remainder as the magnitude taken from the first to that taken from the other.*

Or. " the excess of the first above the third to that of the second above the fourth, as the third to the fourth.—Hose.

$$\because \quad A-C : B-D = A : B; \text{ and } C : D = A : B;$$
$$\therefore A-C : B-D = C : D.$$

Cor. 2. *If any magnitudes,* A, B, C, D &c. *are in Geometrical Progression,* i.e., A : B : C : D &c., *the differences,* A ∼ B, : ∼ B : C CD &c,— *will form a geom. progression,* A ∼ B : B ∼ C : C ∼ D &c.—*the successive terms of which have the same ratio with the successive terms of the former.*

D. 1	H.	$\because$ B : C = A : B, & C : D = B : C &c.
2	19, V.	$\therefore$ A ∼ B : B ∼ C = A : B,
		& B ∼ C : C ∼ D = B : C,
		i.e., as A to B, *i. e.* as A ∼ B to B ∼ C; and so on.

Cor. 3. *And conversely, any number of Magnitudes,,* A, B, C, D &c. *in geometrical progression,* A : B : C : D &c.,, *may be considered as the differences of other magnitudes* A, B′ C′ D′ &c., *forming a geometrical progression in which the first term* A′ *is to* A as A to A ∼ B, A′ *to* B′ *as* A *to* B &c., *and the successive terms have the same ratio with the successive terms of the former.*

For let a progression be taken in which A′ : A = A : A ∼ B, and A′ : B′ = A : B.

D. 1	H. Cor. 1, 17, V.	Then $\because A' : B' = A : B$, $\therefore$ *convert.* $A' : A' \sim B' = A : A \sim B$;
2	H. 9, V.	but $A' : A = A : A \sim B$, $\therefore A' : A' \sim B' = A' : A$.
3	D. 2. 14, V.	and $\because$ the 1st A' is the same with the third A', $\therefore A' \sim B' = A$.
4	Cor. 2, 19, V.	but $A' \sim B', B' \sim C', C' \sim D'$ &c., form a progression in which $A' \sim B' : B' \sim C' = A' : B'$, *i. e.* $= A : B$;
5	14, V.	$\therefore B' \sim C' = B, C' \sim D' = C$ &c.
6	Remk.	But the progressions A, B, C, D &c. and $A' \sim B', B' \sim C', C' \sim D', D' \sim E'$ &c. have the same first terms and the same common ratio.
7	Conc.	$\therefore$ those progressions cannot but be identical. *Geom. Pl. Sol. & Sph. p.* 42, 53.

Alg. & Arith. Hyp. Let $a = 15$, $b = 10$ be two Ms; $k = 6$ & $y = 4$ the respective parts.

Alg. By Hyp. $\frac{a}{b} = \frac{x}{y}$;

$\therefore$ $a\,y = b\,x$.

we have to prove that $\frac{a-x}{b-y} = \frac{a}{b}$,

clearing fractions, $ab - bx = ab - by$;

subtract ab, $\therefore bx = ay$ as before.

Because of this identity of products the quantities must be in proportion;

$\therefore a-x : b-y = a : b$.

Cor. Let $\frac{a}{b} = \frac{x}{y}$, $\therefore a\,y = b\,x$ (1).

To prove that $\frac{a}{a-b} = \frac{x}{x-y}$;

clearing fractions $a\,x - a\,y = ax - bx$;

Subtract ax, $ay = b\,x$, an identity with (1);

$\therefore a : a-b = x : x-y$.

Arith. $\frac{15}{10} = \frac{6}{4}$;

$\therefore 15 \times 4 = 10 \times 6$.

to be proved $\frac{15-6}{10-4} = \frac{15}{10}$,

clearing, $(15 \times 10) - (6 \times 10) = (10 \times 15) - (4 \times 15)$,
take away 15×10 $\therefore$ $6 \times 10 = 4 \times 15$.
Now 6×10 & 4×15 give identical products.
$\therefore$ $15 - 6 : 10 - 4 = 15 : 10$.
i. e., $9 : 6 = 15 : 10$.

Let $\frac{15}{10} = \frac{6}{4}$, $\therefore 15 \times 4 = 10 \times 6$ (1).

To prove that $\frac{15}{15-10} = \frac{6}{6-4}$;
clearing, $(16 \times 6) - (15 \times 4) = (15 \times 6) - (10 \times 6)$;
Subtract 15×6, and $15 \times 4 = 10 \times 6$, an identity;
$\therefore$ $15 : 15-10 = 6 : 6-4$. *i. e.* $15 : 5 = 6 : 2$.

PROP. E.—THEOR.

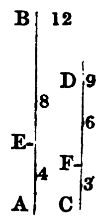
Convertendo. *If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.*

"If four magnitudes be proportional, the first being greater than the second, and the third than the fourth; then the first shall be to its excess above the second, as the third is to its excess above the fourth."—HOSE.

"The terms of an analogy are proportional by conversion."—BELL.

DEM. 17, V. dividendo; B, V. invertendo; 18, V. componendo.

E. 1	Hyp.	Let AB : BE = CD : DF;
2	Conc.	then BA : AE = DC : CF.
D. 1	H.	∵ AB : BE = CD : EF, AB being > BE, & CD > EF,
2	17, V.	∴ *div.* AE : EB = CF : FD.
3	Pr. B, V.	and *invert* BE : EA = DF : CF,
4	18, V.	wherefore *comp.* BE + AE : AE = FD + CF : CF,
		i. e. BA : AE = DC : CF.
5	Rec.	*If therefore four magnitudes, &c.* Q. E. D.



Or, If A : B = C : D, by conversion, A : A — B = C : C — D.

D. 1	H. & 17, V.	∵ A : B = C : D; ∴ *div.* A — B : B = C — D : D.
2	Pr. B, V.	and *inv.* B : A — B = D : C — D;
3	18, V.	∴ *comp.* A : A — B = C : C — D.

Alg. & Arith. Hyp. Let a 12 : b 9 = c 8 : d 6, then a 12 : $a \pm b$ 12 ± 9 = c 8 : $c \pm d$, 8 ± 6.

Alg. $\because \frac{a}{b} = \frac{c}{d}$, and by comp. & divid. $\frac{a \pm b}{d} = \frac{c \pm d}{d}$;

by inverting the fractions, $\frac{b}{a \pm b} = \frac{d}{c \pm d}$;

multiply one side by $\frac{a}{b}$, the other by $\frac{c}{d}$;

then, $\frac{ab}{ab \pm b^2} = \frac{a}{a \pm b}$ & $\frac{cd}{cd \pm d^2} = \frac{c}{c \pm d}$;

$\therefore a : a \pm b = c : c \pm d.$

Arith. $\because \frac{12}{9} = \frac{8}{6}$; comp. and div. $\frac{12 \pm 9}{9} = \frac{8 \pm 6}{6}$,

invert. $\frac{9}{12 \pm 9} = \frac{6}{8 \pm 6}$,

multiply one side by $\frac{12}{9}$, the other by $\frac{8}{6}$,

then $\frac{12}{12 \pm 9} = \frac{8}{8 \pm 6}$;

$\therefore \quad 12 : 12 \pm 9 = 8 : 8 \pm 6.$

Use & App. Among other results, Prop. E., bk. V., leads to the following

1. *If any number of Ms be in continued proportion, as* A : B : C : D : E, *the difference between the first and second terms,* A ∾ B, *is to the first* A, *as the difference between the first and last,* A ∾ E, *is to the sum of all the terms, except the last,* A + B + C + D.

D. 1.	H. & Def. 10, V.	∵ A : B = B : C; B : C = C : D and C : D = D : E.
2	12, V.	∴ A : B = A + B + C + D : B + C + D + E.
3	Pr. E, 5.	∴ *Conv.* A : A ∾ B = A + B + C + D : A ∾ E; B being > A and D > C;
4		And ∵ (A + B + C + D) — (B + C + D + E) = A ∾ E,
5	P. B. V.	∴ A ∾ B : A = A ∾ E : A + B + C + D.

2. *In a series of continued proportionals*, A : B : C : D : E, &c., *the differences of the successive terms* A $\sim$ B, B $\sim$ C, C $\sim$ D, &c., *are also in continued proportion*,— A $\sim$ B : B $\sim$ C : C $\sim$ D, and B $\sim$ C : C $\sim$ D = C $\sim$ D : D $\sim$ E.

CASE I. *Let* A > B,—*the series is continually decreasing*; then A — B : B — C = B — C : C — D, and B — C : C — D = C — D : D — E.

D. 1.	H. 17, V.	∵ A : B = B : C, ∴ *div.* A — B : B = B — C : C;
2	16, V.	and *altern.* A — B : B — C = B : C,
3	*Sim.*	So, ∵ B : C = C : D, ∴ B—C : C—D = C : D.
4	H. 11, V.	but B : C = C : D, ∴ A — B : B — C = B — C : C — D.
5	*Sim.*	So B — C : C — D = C — D : D — E;
6	Rec.	∴ A — B : B — C : C — D : D — E, are continued proportionals.

CASE II. If A < B, *the series is continually increasing.*

By a like method this case is also proved.

3. *In an infinitely decreasing series of Magnitudes in continued proportion, the first term*, A, *is a mean proportional between its excess above the second*, B, *and the sum of the series.*

Let A, B, denote the 1st and 2nd terms,— Z the last term, and S the sum of the Series, then, as in the last example but one, Use 1, E, V., A — B : A = Z : S — Z. But the last term Z may be less than any magnitude or quantity we fix on, however small; and hence to the values of A — Z and S — Z there will be limits namely A and S.

∴ A — B : A = A : S.

PROP XX—THEOR..

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first be greater than the third, the fourth shall be greater than the sixth, and if equal, equal; and if less, less.

DEM. 8, V. The gr. M a gr. ratio.

13, V. If 1st : 2nd = 3rd : 4th, but 3rd : 4th > 5th : 6th, the 1st : 2nd > 5th : 6th.

Cor. 13, V. If 1st : 2nd > 3rd : 4th, but 3rd : 4th = 5th : 6th, the 1st : 2nd > 5th : 6th.

10, V. The M with gr. ratio the gr. of two Ms.

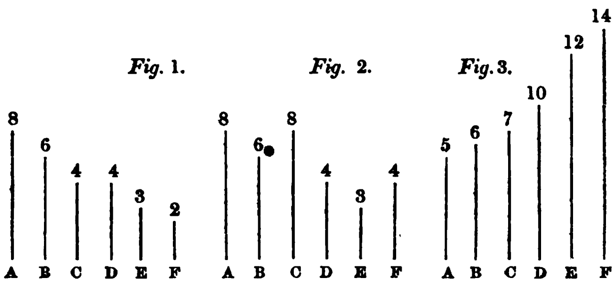
7, V. Equal Ms. the same ratio to the same M &c.

11, V. Ratios the same to the same r. the same to one another.

Pr. B, V. *Invertendo.* 9, V. Ms. with the same ratio equal &c.

E.	1	Hyp. 1.	Let there be three Ms. A, B, C, & three other Ms. D, E, F ;
	2	Hyp. 2.	let A : B = D : E, & B : C = E : F ;
	3	Conc.	if A > = or < C, then D > = or < F.

Fig. 1. *Fig.* 2. *Fig.* 3.



Case I. *Let* A > C, *then* D > F. *Fig.* 1.

D.	1	H.	∵ A > C, & B is another M;
	2	8, V.	∴ A : B > C : B.
	3	H & 13, V.	but D : E = A:B, ∴ D : E > C : B.
	4	H & B, V.	And ∵ B : C = E : F, ∴ *inv.* C : B = F:E ;
	5	D. 3, Cor. 13, V.	and ∵ D : E > C : B, ∴ D : E > F : E ;
	6	10, V.	∴ D > F.

A.	B.	C.
D.	E.	F.

Case II. *Let* A = C, *then* D = F. *Fig.* 2.

D.	1	H. & 7, V.	∵ A = C, ∴ A : B = C : B;
	2	H.	but A : B = D : E, & C : B = F : E ;
	3	11, V. B, V. 9, V.	∴ D : E = F : E, & ∴ D = F.

CASE III. *Let* A < C, then *then* D < F. *Fig.* 3.

D. 1	H. & Case 1.	∵ $C > A$, and $C : B = F : E$, & $B : A = E : D$;
2	Case 1.	∴ $F > D$, *i. e.*, $D < F$.
3	Rec.	Therefore, *if there be three magnitudes*, &c. Q. E. D.

Arith. Hyp. Let the Magnitudes be represented by numbers, *Case* 1. by 8, 6, 4, 4, 3. 2 ; *Case* 2. by 8, 6, 8, 4, 3, 4 ; and *Case* 3. by 5, 6, 7, 10, 12, 14.

Case 1. If $\frac{8}{6} = \frac{4}{3}$, and $\frac{6}{4} = \frac{3}{2}$; then the 1st, 8 > 3rd, 4, & 5th, 4 > 6th 2

Case 2. If $\frac{8}{6} = \frac{4}{3}$, and $\frac{6}{8} = \frac{3}{4}$; then the 1st, 8=3rd 8, & 4th 4=6th 4.

Case 3. If $\frac{5}{6} = \frac{10}{12}$, and $\frac{6}{7} = \frac{12}{14}$; then the 1st 5 < 3rd, 7, & 4th 10 < 6th 14.

SCH. The Proposition is also ennunciated, with reference to the formula annexed, "If the first magnitude, A, be to the second B, as the third, C, is to the fourth, D ; and, if the second B, be to the fifth E, as the fourth, D, is to the sixth, F ; then the third, C, shall be greater than, equal to, or less than the sixth, F, according as the first, A, is greater than, equal to, or less than the fifth E."—HOSE.

A.	C.
B.	D.
E.	F.

There are of course *three* Cases, A > E, A = E, and A < E, to be proved as in the foregoing proposition from SIMSON's text.

PROP. XXI.—THEOR.

If there be three magnitudes and other three which have the same ratio taken two and two, but in a cross order, i. e. in proportione perturbatâ, *in disturbed proportion; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth*; *and if equal, equal; and if less, less.*

DEM. 8, V. 13, V. Pr. B, V. *invertendo;* Cor. 13, V. 10, V. 7, V. 11, V. and 9, V.

E.	1	Hyp. 1	Let there be two series of Ms, A, B, C and D, E, F :
	2	2.	and let A : B = E : F, and B : C = D : E;
	3	Conc.	then, if A >, =, or < C, D >, =, or < F.

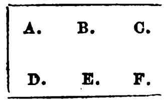
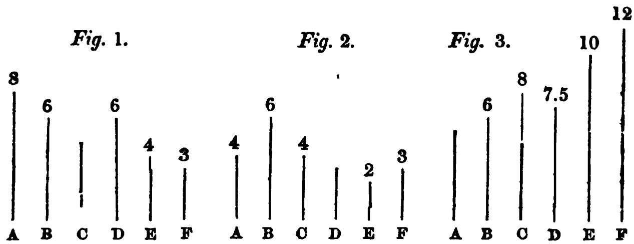


Fig. 1. *Fig.* 2. *Fig.* 3.



Case. I. *Let* A > C, *then* D > F. *Fig.* 1.

D.	1	H. & 8, V.	∴ A > C, and B another M, ∴ A : B > C : B;
	2	H. & 13, V.	but E : F = A : B, ∴ E : F > C :B;
	3	H. Pr. B, V.	& ∵ B : C=D : E, ∴ *inv.* C : B =E : D;
	4	D. 2. Cor. 13, V.	& ∵ E : F > C : B, ∴ E : F > E : D;
	5	10, V.	∴ F < D, *i.e.*, D > F.

Case II. *Let* A = C, *then* D = F. *Fig.* 2.

D.	1	H & 7, V.	∵ A = C, ∴ A : B = C : B;
	2	H. D.3, Case I.	but A : B = E : F, & C : B = E : D,
	3	11, V.	∴ E : F = E : D;
	4	9, V.	and ∴ D = F.

Case III. *Let* A < C, *then* D < F. *Fig.* 3.

D.	1	H & D. 3, Case I.	∵ C > A, ∴ C : B = E : D,
	2		and B : A = F : E;
	3	Case I.	∴ F > D, *i. e.* D < F.
	4	Rec.	Therefore, *if there be three magnitudes, &c.*

Q. E. D.

Arith. Hyp. Let the two series of three magnitudes each be represented by numbers,—*Case* 1. by 8, 6, 4, 6. 4, 3; *Case* 2. by 4, 6, 4, 3, 2, 3: and *Case* 3. by 5, 6, 8, 7.5, 10, 12.

Case 1. If $\frac{8}{6} = \frac{4}{3}$, & $\frac{6}{4} = \frac{6}{4}$; then if 1st 8 $>$ 3rd 4, the 4th 6 $>$ 6th, 3.

Case 2. If $\frac{4}{6} = \frac{2}{3}$ & $\frac{6}{4} = \frac{3}{2}$; then if 1st 4 $=$ 3rd 4, the 4th 3 $=$ 6th, 3.

Case 3. If $\frac{5}{6} = \frac{10}{12}$, & $\frac{6}{8} = \frac{7 \cdot 5}{10}$; then if 1st 5 $<$ 3rd 8, the 4th 7.5 $<$ 6th 8.

Sch. The following is a variation of Prop. 21, having reference to the annexed formula; "If the first magnitude, A, be to the second B, as the third C is to the fourth D; and if the second, B, be to the fifth, E; as the sixth, F, to the third, C; then the sixth, F, shall be greater than, equal to, or less than the fourth, D, according as the first, A, is greater than, equal to, or less than the fifth E."—Hose.

A.	C.
B.	D.
E.	F.

Again, there are *three* Cases, A $>$ E, A $=$ E, & A $<$ E; and the demonstration depends on the same principles as those employed in Simson's Text.

Prop. XXII.—Theor.

Ex æquali, *or* ex æquo, by equality. *If there be any number of magnitudes and as many others, which taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others.*

'If there be any number of magnitudes, and as many others, which taken two and two, have the same ratio, they shall have the same ratio by equality."—Euclid.

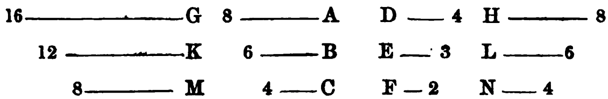
Con. Pst. 1, V. Dem. 4, V. In an analogy equims. of 1st and 3rd have the same ratio to any equims. of 2nd and 4th.

20, V. In the two series of 3 Ms, which taken two and two have the same ratio, of the 1st $>$ $=$ or $<$ 3rd, the 4th $>$ $=$ or $<$ 6th.

Def. 5, V. Criterion of the equality of ratios.

CASE I. *Let there be two series of three Ms, A, B, C, and D, E, F.*

E. 1	Hyp.	Let A : B = D : E, & B : C = E : F;
2	Conc.	then A : C = D : F.



C. 1.	Pst. 1, V.	Of A,D take equims G,H; of B,E equims K,L; and of C,F, equims. M,N.
D. 1	H & C.	∵ A : B = D : E; & G,H equims of A,D, and K,L, equims of B,E;
2	4, V	∴ G : K = H : L; & K : M = L : N;
3	D. 2.	and ∵ G ,K,M are 3 mags.. & H,L,N, 3 others, which taken two and two have the same ratio;
4	20, V.	∴ if G >, = or < M, H >, =, or < N.
5	C.	But G,H are equims. of A,D; M,N equims of C,F;
7	Def. 5, V.	∴ A : C = D : F.

CASE II, *Let there be two series of four Ms each,* A,B,C,D *and* E,F,G,H.

E 1.	Hyp.	Let A : B = E : F; B : C = F : G, and C : D = G : H;
2	Conc.	then A : D = E : H.

A. 12 — B. 6 — C. 18 — D. 36.
E. 6 — F. 3 — G 9. — H. 18.
Then A 12 : D 36 = E 6 : H 18.

D. 1	H.	∵ A, B, C, are 3 Ms, and E, F, G, three others, which taken two and two have the same ratio;
2	Case 1,	∴ A : C = E : G;
3	H. & Case 1,	but C : D = G : H, ∴ A : D = E : H.
4	*Sim,*	And in a similar way, whatever be the number of Ms.
5	Rec.	Therefore, *if there be any number &c.*

Q. E. D.

Alg. & Arith. Hyp. Take the two series of quantities, a 12, b 6, c 18, and a' 6, b' 3, c' 9.

Alg. By Hyp. $\frac{a}{b} = \frac{a'}{b'}$, and $\frac{b}{c} = \frac{b'}{c'}$;

Multiply. $\frac{a}{b} \times \frac{b}{c}$, and $\frac{a'}{b'} \times \frac{b'}{c'}$;

We have $\frac{a\,b}{b\,c}$, or $\frac{a}{c} = \frac{a'\,b'}{b'\,c'}$, or $\frac{a'}{c'}$;

i. e. $a : c = a' : c'$.

Arith. By H. $\frac{12}{6} = \frac{6}{3}$, & $\frac{6}{18} = \frac{3}{9}$;

Mult. $\frac{12}{6} \times \frac{6}{18}$, & $\frac{6}{3} \times \frac{3}{9}$;

We have $\frac{72}{108}$, or $\frac{12}{18}$, & $\frac{18}{27}$, or $\frac{6}{9}$.

i. e. $12 : 18 = 6 : 9$.

COR. Whatever be the number of Analogies,— for instance, three, *i. e.* A : B = C : D;, B : E = D : F, and E : G = F : H; if they are so constituted that the second and fourth terms of each, as B & D, form respectively the first and third of the next, as B & D; then A, the first term of the next proportion shall be to the second of the last, as the third C of the first proportion to the fourth of the last.

A2	: $\cancel{\text{B3}}$	=	C4	: $\cancel{\text{D6}}$
$\cancel{\text{B3}}$	: $\cancel{\text{E5}}$	=	$\cancel{\text{D6}}$	: $\cancel{\text{F10}}$
$\cancel{\text{E5}}$	: $\cancel{\text{G4}}$	=	$\cancel{\text{F10}}$	: $\cancel{\text{H8}}$
$\cancel{\text{G4}}$	: $\cancel{\text{I12}}$	=	$\cancel{\text{H8}}$	: $\cancel{\text{K24}}$
$\cancel{\text{I12}}$	: L9	=	$\cancel{\text{K24}}$	: M18
A2	: L9	=	C4	: M18

In other words,—" *Ratios compounded of any number of equal ratios in the same order, are equal to one another.*"

For, by Hyp. $\because$ A : B = C : D, and B : E = D : F,

$\therefore$ A : E = C : F.

And again, $\because$ A : E = C : F, and E : G = F : H;

$\therefore$ A : G = C : H.

And so on, whatever be the number of proportions.

SCH. The 22nd Prop. may be thus varied ; "If the first Mag. A, be to the second B, as the third C, is to the fourth D ; and if the second Mag. B, be to the fifth E, as the fourth D is to the sixth F ; then the first A shall be to the fifth E, as the third C, is to the sixth F."—HOSE.

A similar process of reasoning is to be followed. as in Case 1 and 2, Pr. 32

G.	A.	C.	H.
K.	B.	D.	L.
M.	E.	F.	N.

USE AND APP. By combining the principles contained in Props. 18, 17 and 22, *i. e. componendo.* and *ex æquo*, we arrive at the further truth, that

Proportionals remain proportional miscendo, *by mixing*, or as it is sometimes named,—by using the sum and difference.

E. 1	Hyp.	Let $A : B = C : D$;
2	Conc.	then $A + B : A \sim B = C + D : C \sim D$;
		adopting $A - B$, or $B - A$ and $C - D$ or $D - C$,
		as $B <$ or $>$ A, and $C <$ or $>$ D.
D. 1	18, V.	*Comp.* $A + B : B = C + D : D$;
2	17, V.	*Div.* $A - B : B = C - D : D$;
3	Pr. B, V.	*Inv.* $B : A - B = D : C - D$;
4	D. 1.	but $A + B : B = C + D : D$;
5	22, V.	*Ex æquo* $A + B : A - B = C + D : C - D$.

PROP. XXIII.—THEOR.

Ex æquo perturbato. *If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio ; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others.*

"If there be any number of magnitudes, and as many others, which taken, two and two, have the same ratio, and their proportion be disturbed, they shall be in the same ratio by equality."—EUCLID.

N. B. This Proposition is usually cited by the words, "*ex æquali in proportione perturbatâ*, by equality in perturbate proportion; or "*ex æquo perturbato*, by perturbate equality.

CON. Pst. 1, V. DEM. 15, V. *Ms* have the same ratio as their equims.

11, V. Ratios the same to the same ratio, the same to one another.

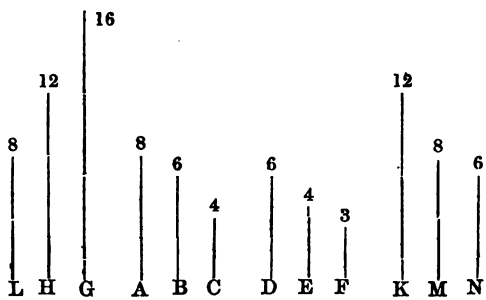
4, V. Equims of the 1st and 3rd have the same ratio to equims of 2nd and 4th,

21, V, Two series of 3 Ms. in each having the same ratio but in a cross order.

Def. 5, V. Criterion of the equality of ratios.

CASE I. *Let there be two series of* 3 *Ms, each,* A,B,C,D,E,F, *which taken two and two in cross order have the same ratio.*

E.	1	Hyp.	Let A : B = E : F, & B : C = D : E ;
	2	Conc.	then A : C = D : F.
C.	1	Pst. 1, V.	Take of A,B, D any equims. G,H,K ;
	2	,,	and of C,E,F any equims. L,M,N.



D.	1	C. & 15, V.	∵ G,H are equims. of A,B; ∴ A : B = G : H,
	2	*Sim.*	∵ M,N are equims of E,F, ∴ E : F = M : N;
	3	H & 11, V.	but A : B = E : F, ∴ G : H = M : N;
	4	H.	And ∵ B : C = D : E, & H,K are equims of B,D, & L,M equims. of C,E ;
	5	4, V.	∴ H : L = K : M;
	6	D. 3.	now G : H = M : N, and G,H,L — K,M,N, are two series of 3 Ms. each, and taken in cross order two and two, they have the same ratio ;
	7	21, V.	∴ if G >, =, or < L, K >, = or < N ;
	8	C. 1 & 2.	but G,K, any equims. of A,D. & L,N any equims. of C, F ;
	9	Def. 5, V.	∴ A : C = D : F.

CASE II. Next, *let there be two series of four Ms. each,* A, B, C, D, and E, F, G, H *which taken two and two in a cross order also have the same ratio* ;

E. 2	Hyp.	And let A : B = G : H; B : C = F : G; and C : D = E : F;
2	Conc.	then A : D = E : H.

A 2	B 3	C 4	D 12
E 5	F 15	G 20	H 30

A 2	: B 3	= G 20	:	H 30
B 3	: C 4	= F 15	:	G 20
C 4	: D 12	= E 5	:	F 15
A 2	: D 12	= E 5	:	H 30

D. 1	H.	∵ A,B,C & F,G,H, are two series of 3 Ms each, and taken 2 & 2 in a cross order have the same ratio.
2	Case 1.	∴ A : C = F : H;
3	H. & Case 1.	but C : D = E : F, ∴ A : D = E : H.
4	*Sim.*	And so on, whatever be the no. of *Ms.*
5	Rec.	Therefore, *if there be any number*, &c. Q. E. D.

Alg. & Arith. Hyp. Let $a = 8, b = 6, c = 4; \ a' = 6, b' = 4, c' = 3.$

Alg. By Hyp. $\frac{a}{b} = \frac{b'}{c'}$, and $\frac{b}{c} = \frac{a'}{b'}$;

Multiply ; $\frac{a\,b}{b\,c} = \frac{b'\,a'}{c'\,b'}$, or $\frac{a}{c} = \frac{a'}{b'}$.

i. e. $a = c = a' : c'$.

Arith. $\frac{8}{6} = \frac{4}{3}$, & $\frac{6}{4} = \frac{6}{4}$;

$\frac{8 \times 6}{6 \times 4} = \frac{4 \times 6}{3 \times 4}$, or $\frac{8}{4} = \frac{6}{3}$;

i. e. 8 : 4 = 6 : 3.

Illustration. If A 12 : B 6 C 3 : D 9,
E 4 : F 12 G 6 : H 3,

then, A 12 : D 9 E 4 : H 3.

SCH. There are different ways of announcing Prop. 23.

1. Ratios, compounded of any number of equal ratios, but in reverse order, are equal to one another;

For let there be the two series, A,B,C,D and E,F,G,H;
As before, Pr. 23 Case 2, if $A : B = G : H$,
$B : C = F : G$,
and $C : D = E : F$;
then, *ex æquo perturbato*, $A : D = E : H$.

2. In any two Series of Magnitudes, A, B, C & D, E, F, which taken two and two in a cross order, have the same ratio, if A, the first M, be to B, the second, as C the third is to D the fourth; and if B, the second, be to E the fifth, as F the sixth, to C the third; then the first A is to the fifth E, as the sixth F to the fourth D.

And if there be any number of Analogies; for instance three: and if the second and third terms of each Analogy form respectively the first and fourth terms of the next; then the first term, A of the first Analogy, shall be to the second, G, of the last, as the third term H, of the last, is to D the fourth of the first.

A 12	: B 6	=	C 6	: D 3
B 6	: E 4	=	F 9	: C 6
E 4	: G 2	=	H 18	: F 9
H 12	: G 2	=	H 18	: D 3

The same conclusion would follow were the number of Analogies four or more.

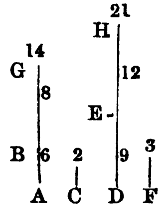
PROP. XXIV.—THEOR.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

DEM. Pr. B. V. Four Ms being proportionals are proportionals *inversely.* 22, V. *Ex æquo.* 18, V. *Componendo.*

Pr, A, V. If 1st : 2nd = 3rd 4th,—then if 1st $> =$ or $<$ 2nd, the 3rd $> =$ or $<$ 4th. 17, V. *Dividendo.*

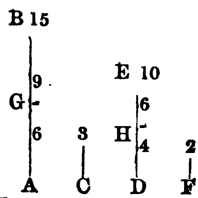
E. 1	Hyp.	Let AB : C=DE : F, and BG : C=EH : F;
2	Conc.	then AB+BC, *i. e.*, A G : C=DE+EH, *i. e.* DH : F.
D. 1	H. & Pr. B, V.	∵ BG : C=EH : F, and *inv.* C : BG = F : EH;
2	H. & D. 1.	and ∵ AB : C=DE : F, and C : BG=F : EH;
3	22, V.	∴ *ex æq.* AB : BG = DE : EH;
4	18, V.	∴ *comp.* AG : BG=DH : HE;
5	H. & 22, V.	but GB : C=HE : F; ∴ *ex æq.* AG : C = DH : F.
6	Rec.	∴ If *the first has to the second, &c.* Q. E. D.



Arith. Illust. If A B 6 D E 9.
C 2, F 3.
B G 8. E F 12.

Then A B + B G : C = D E + E H : F;
Or, A G 14 : C 2 = D H 21 : F 3.

Cor. 1. On the same Hypothesis being made, the first however, being greater than the fifth and the third than the sixth, it follows, that "*the excess,* A G, *of the first,* AB, *above the fifth,* BG, *shall be to the second* C, *as the excess,* DH, *of the third* D E, *above the sixth,* E H, *is the fourth* F; or, in other words, the difference between the first and fifth shall be to the second, as the difference between the third and sixth is to the fourth.



E. 1	Hyp.	Let AB : C = DE : F, and BG : C = EH : F, AB being > BG;
2	D. 3, Pr. A. V.	and ∴ AB : BG = DE : EH, & DE>EH;

3	Conc.	then AG, *i. e.* $AB - BG : C = DH$, *i. e.* $DE - EH : F$.
D. 1	H.	$\because AB : BG = DE : EH$, AB being $> BG$ and $DE > EH$;
2	17, V.	$\therefore$ *div.* $AG : BG = DH : EH$;
3	H. & 22, V.	and $\because BG : C = EH : F$; $\therefore$ *ex. æq.* $AG : C = DH : F$.
4	Rec.	$\therefore$ *the excess of the first, &c.* Q. E. D.

Cor. 2. The Proposition holds true of two ranks of magnitudes whatever be their number, *of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to the fourth magnitude*; Or, in any number of Proportions, if the second term, B, is the same throughout, and also the fourth term, D, the same; then the sum of all the first terms, $A + E + G$, is to the common second term, B, as the sum of all the third terms, $C + F + H$, is to the common fourth term, D.

$$\begin{array}{c} A\,4 : B\,2 = C\,6 : D\,3, \\ E\,6 : B\,2 = F\,12 : D\,3, \\ G\,10 : B\,2 = H\,15 : D\,3. \\ \hline A+E+G\;22 : B\,2 = C+F+H\,33 : D\,3. \end{array}$$

E. 1	Hyp.	Suppose three proportions so constituted, that $A : B = C : D$; $E : B = F : D$, and $G : B = H : D$;
2	Conc.	then $A + E + G : B = C + F + H : D$.
D. 1	H.	$\because A : B = C : D$, and $E : B = F : D$;
2	18, V.	$\therefore A + E : B = C + F : D$.
3	D. 2, & H.	And $\because A + E : B = C + F : D$, and $G : B = H : D$;
4	18, V.	$\therefore A + E + G : B = C + F + H : D$.
5	*Sim.*	In the same way for any number of proportions.

SCH. Prop. 24 may also be expressed,—"*If two series of Proportionals have the same consequents*, B, D, *the sum of the first antecedents*, $A + E$, *shall be to their common consequent*, B, *as the sum of the second antecedents*, $C + F$, *to their common consequent*, D.

E. 1	Hyp.	Let $A : B = C : D$, and $E : B = F : D$;
2	Conc.	Then $A + E : B = C + F : D$.
D. 1	H.	$\because E : B = F : D$, $\therefore$ *inv.* $B : E = D : F$;
2	H. & 22, V.	but $A : B = C : D$, $\therefore$ *ex. æq.* $A : E = C : F$;
3	18, V.	and *comp.* $A + E : E = C + F : F$.
4	H.	Again, $E : B = F : D$;
5	22, V.	$\therefore$ *ex æquo* $A + E : B = C + F : D$.

COR. 1. Thus also by *dividendo* instead of *componendo*,—"*If two proportions have the same consequents*, the difference *of the first antecedents shall be to their common consequent, as* the difference *of the second antecedents to their common consequent.*

E. 1	Hyp.	Let $A : B = C : D$, and $E : B = F : D$;
2	Conc.	then $A-E : B = C - F : D$.

COR. 2. *And if four magnitudes form a proportion*, $A\,10 : B\,5, = C\,6 : D3$, *then miscendo*, using the sum and difference, as in Use & App. Pr. 22, V., *the sum of the first and second*, $A + B$, *is to their difference*, $A \sim B$, *as the sum of the third and fourth*, $C + D$ *to their difference* $C - D$.

D. 1	18. V.	*comp.* $A + B : B = C + D : D$:
2	17, V.	*div.* $A - B : B = C - D : D$;
3	B. V.	*inv.* $B : A - B = D : C - D$;
4	22, V.	*ex æq.* $A + B : A - B = C + D : C—D$.

COR. 3. In any number of magnitudes of the same kind forming two series, one A, B, C, D, E, F, &c., and the other as many, G, H, I, K, L, M, &c. —*if the ratios of the first to the second*, $A : B$, or $G : H$; *of the second to the third*, $B : C$, or $H : I$; *of the third to the fourth*, $C : D$, or $I : K$; *and so on, be the same in the two series; then, any two combinations whatever*, miscendo, i. e., *by the sum and difference of the magnitudes of the first series*, $A + C - E$, and $B - C + D$, *shall be to one another as two similar combinations of the corresponding magnitudes of the second series*, $G + I - L$, and $H - I + K$. See *Geom. Plane, Sol. & Sph. p.* 55.

Prop. XXV.—Theor.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Con. 2 & 3, I. To draw a line equal to a given st. line, and from a gr. line to cut off a part equal to the less.

Dem. 11, V. Ratios the same to the same ratio are the same to one another.

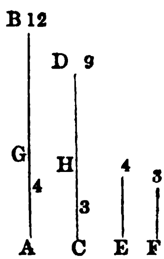
7, V. Equal Ms the same ratio to the same M, and conversely.

19, V. If a M : a M = a pt. : a pt.—the rem : the rem. = the M : the M.

A, V. If 1st : 2nd = 3rd : 4th, then, if 1st > = or < 2nd, 3rd, > = or < 4th.

Ax. 2. I. If equals be added to equals, the wholes are equal.

Ax. 4, I, If equals be added to unequals the wholes are unequal.



E. 1	Hyp.	Let AB : CD = E : F;
2	14, V. & A, V.	and let A B be the greatest, consequently F the least;
3	Conc.	then AB + F > CD + E.
C.	2 & 3, I.	Take in AB, AG = E, and in CD, CH = F.
D. 1	H. & C.	∵ AB : CD = E : F, and AG=E, & CH=F;
2	11 & 7, V.	∴ AB : CD = AG : CH.
3	D. 2.	but ∵ the whole AB : the whole CD = pt. AG : pt. : CH;
4	19, V.	the rem. GB : rem. HD = AB : CD.
5	H. & A, V.	but AB > CD, ∴ GB > HD.
6	C.	And ∵ AG = E, and CH = F;
7	Ax. 2, I.	∴ AG + F = CH + E.
8	*Add.*	To the unequal Ms, BG, HD, BG being > HD, add AG+F, and CH+E respectively.
9	Ax. 4, I.	∴ BG + AG + F > HD + CH + E, *i. e.*, AB + F > CD + E.
10	Rec.	Therefore, *if four magnitudes of the same kind, &c.* Q. E. D.

Arith. Illust. If AB 12 : CD 9 = E 4 : F 3;
Then A + F, 12 + 3 > CD + E, 9 + 4.

N.B. This Proposition has been inserted as Corollary 2 to Pr. 17, bk. V.

Cor. *If three magnitudes be proportionals,* A : B: C, *the sum of the extremes* A + C, *will be greater than twice the mean,* 2 B, *and therefore half the sum* $\frac{A+C}{2}$ *greater than the mean* B.

For the proof see Cor. 3, Pr. 17, bk. V.

Thus the *arithmetical mean* between two magnitudes *is greater* than the *geometrical mean* between them, the case excepted in which the magnitudes are equal to one another.

SUPPLEMENTARY PROPOSITIONS.

It has been customary from very early times, perhaps from the days of Euclid himself, to add several Propositions to this Fifth Book. The first English Edition, which added *Nine*, announces them in this way,—"Here follow certayne propositions added by *Campane* which are not to be contemned, and are cited even of the best learned, namely of *Johannes Regio Montanus*, in the Epitome which he writeth vpon Ptolome."—Billingsley's Euclid, fol. 150.

The Supplementary Propositions here given are chiefly from Simson with some from other sources.

Prop. F.—Theor.

Ratios which are compounded of the same ratios are the same to one another.

Dem. 22. V. *Ex æquo* by equality. 23, V. *Ex æquo perturbato* by perturbate equality.

E.	1	Hyp.	Let A : B = D : E & B : C = E : F;
	2	Conc.	then the ratio compounded of A : B & B : C,—(by Def. A, V.) the ratio A : C,— shall be the same with the ratio D : F, which (by Def. A, V.) is compounded of D : E & E ; F.
D.	1	H.	∵ in the 2 series of Ms, A,B,C, & D,E,F, A : B = D : E, & B : C = E : F;

A 6.	B 4.	C. 10.
D 3.	E 2.	F. 5.
A 6 : C 10=D 3 : F 5		

A 8.	B 6.	C 8.
D 3.	E 4.	F 3.
A 8 : C : 8=D 3 : F 3		

	2	22, V.	∴ *ex æq.* A : C = D : F.
	3	H.	Next, ∵ A : B = E : F, and B : C = D : E;
	4	23, V.	∴ *ex æq. pert.* A : C = D : F. *i. e.*, ratio A : C compd. of A : B & B : C is the same with ratio D : F „ D : E & E : F.
	5	*Sim.*	In like manner for any number of ratios.
	6	Rec.	Therefore, *Ratios which are compounded*, &c.

SCH. Two cases only are demonstrated in the above proposition ; *one*, of ratios compounded of the same ratios in the same order, as in Pr. 22, V: the *other* of ratios compounded of the same ratios in a reverse order as in 23, V. There remains the Case of *Ratios compounded in any other order*; which may be demonstrated in a similar way.

"For if K,L,M represent the three ratios in one order, in whatever other order they may be arranged, two of them will be found which are contiguous in both arrangements ; commencing with which two, the demonstration will differ little from the above."

Also, "*ratios which are compounded of the same four ratios,* K,L,M,N *in whatsoever orders, are the same with one another* ; as for instance, in the orders K,L,M,N, and M,K,N,L ;—for the latter ratio is the same with the ratio which is compounded of the same ratios in the order M,K,L,N, because the ratio which is compounded of K,N,L, is the same with that which is compounded of K,L,N ; and for a similar reason, the ratio which is compounded of M,K,L,N, is the same with that which is compounded of K,L,M,N."

"And the same reasoning may be extended to five, six, or any other number of ratios."—GEOM. PLANE, SOL. & SPH. p. 55, 56.

Prop. G.—Theor.

If several ratios be the same to several ratios, each to each; the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

Dem. Def. A, V. In any number of Ms of the same kind, the 1st has to the last the ratio compd. of the ratio which the 1st has to the 2nd, and of the ratio which the 2nd has to the 3rd, and of the ratio which the 3rd has to the 4th, and so on unto the last magnitude.

22; V. *Ex æquo* by equality.

E.	1	Hyp. 1.	Let A : B = E : F, & C : D = G : H; also A : B = K : L, & C : D = L : M;
	2	Conc. 1. by Def. A, V.	then, K : M is compounded of K; L & L : M; and K : L & L : M the same with A : B & C : D;
	3	Hyp. 2.	Again as E : F let N : O, & as G : H let O : P;
	4	Conc. 2, by Def. A, V.	then ratio N : P is compd. of ratios, N : O & O : P; and N : O, & O : P are the same with E : F and G : H;
	5	Conc. 3.	And it is to be shewn that K : M = N : P.

A 6.	B 4.	C 8.	D 6.	K 12.	L 8.	M 6.
E 3.	F 2,	G 4.	H 3.	N 18.	O 12.	P 9.

K 12 : M 6 = 18 : P 9.

D.	1	H. 1 & 2.	∵ K : L, as (A : B, as E : F as) N : O;
	2	H. 1 & 2.	and ∵ as L : M so is (C : D, and so is G : H, and so is) O : P;
	3	22, V.	∴ *ex æquali.* K : M = N : P;
	4	Rec.	Therefore, *if several ratios be the same, &c.* Q. E. D.

Otherwise

E. 1	Hyp. 1.	In two series of Ms, A, B, C, D; and A', B', C', D', let $A : B = A' : B'$; and $C : D = C' : D'$;
2	Hyp. 2.	Also in two other series, K, L, M, and K', L', M', let $K : L = A : B$; and $L : M = C : D$;
3		Also $K' : L' = A' : B'$, and $L' : M' = C' : D'$;
4	Conc. 1.	Then, (by Def. A, V.) ratio $K : M$ is compd. of $K : L$ and of $L : M$; and ratios $K : L$ and $L : M$ are equal to ratios $A : B$ and $C : D$.
5	Conc. 2.	Again, (by Def. A, V.) ratio $K' : M'$ and of $K' : L$ and of $L' : M'$; and ratios $K' : L'$ and $L' : M'$ are equal to ratios $A' : B'$ and $C' : D'$;
6	Conc.	It is then to be proved that ratio $K : M =$ ratio $K' : M'$,
D. 1	H. 2, 1.	$\because K : L = A : B$, and $A : B = A' : B' = K' : L'$;
2	11, V.	$\therefore K : L = K' : L'$.
3	H. 2 & 3.	Again, $\because L : M = C : D$, and $C : D = C' : D'$, and $C' : D' = L' : M'$;
4	11, V.	$\because L : M = L' : M'$.
5	22. V.	$\therefore$ in series K, L, M, and K', L', M', *ex æq.* $K : M = K' : M'$.

Prop. H.—Theor.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them;—then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Dem. Pr. B, V. *Invertendo.* 22, V. *Ex æquo.*

E. 1	Hyp. 1.	Let the first ratios be $A : B$, $B : C$, $C : D$, $D : E$ $E : F$,
2	„ 2.	and the other ratios $G : H$, $H : K$, $K : L$, $L : M$;

3	" 3.	Also let A : F, compd. of the 1st ratios, = G : M, compd. of the other ratios;
4	" 4.	and let A : D, compd. of A : B, B : C, C : D, be the same with the ratio G : K, compd. of G : H, and H : K,
5	Conc.	then the ratio D : F, compd. of D : E, and E : F shall equal K : M, compd. of K : L, and L : M.

A 12	B 8.	C 6.	D 4.	E 3.	F 2.
G 18.	H 12.	K 6.	L 6.	M 3.	

D 4 : F 2 = K 6 : M 3.

D. 1	H. Pr. B, V.	∵ A : D = G : K; ∴ (*inv.*) D : A = K : G.
2	H. & 22, V.	and ∵ A : F = G : M; ∴ *ex æquo.* D : F = K : M.
3	Rec.	Therefore, *if a ratio, which is compounded, &c.* Q. E. D.

Prop. K.—Theor.

If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be

the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then, the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same, each to each, to the remaining ratios of the first, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same each to each, to these remaining ratios.

DEM. Def. A, 5. Compound Ratio. 22, V. *Ex æquo.*

11, V. Ratios the same to the same ratio are the same to one another.

Pr. B, V. *Invertendo.*

E,	1	Hyp. 1.	Let the *first* ratios be A : B, C : D, E : F;
	2	„ 2.	& the *others*, G : H, K : L, M : N, O : P, Q : R.
	3	„ 3.	and let A : B = S : T; C : D = T : V; and E : F = V : X;
	4	Conc.	then (Def. A, V.) S : X is compd. of S : T, T : V, and V : X, which are the same to A : B, C : D, and E : F, each to each.
	5	Hyp. 4.	Also let G : H = Y : Z, and K : L = Z : a; M : N = a : b; O : P = b : c, and Q : R = c : d;
	6	Conc. 2.	then, (Def. A, V.) Y : d is compd. of Y : Z, Z : a, a : b, b : c, and c : d; which are the same, each to each, to the ratios of G : H, K : L, M : N, O : P, and Q : R;
	7	Conc. 3.	∴ S : X = Y : d.
	8	Hyp. 5.	Also, let A : B, *i. e.*, S : T = e : g, compd. of e : f, and f : g, the same as G : H and K : L;
	9	Hyp. 6.	and let h : l be compd. of h : k and k : l; which are the same to the remaining *first* ratios, namely, to those of C : D and E : F,
	10	Hyp.	Also let m : p be compd. of m : n, n : o, & o : p; which are the same, each to each, to the remaining *other* ratios, of M : N, O : P and Q : R.
	11	Conc. 4.	*then* h : l, = m : p.

$h,\ k,\ l.$
4, 14, 10.

A, B; C, D; E, F. S, T, V. X.
18, 9. 8, 28. 21, 15. 4, 2, 7. 5.

G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
9, 6. 12, 9. 6, 9. 15, 20. 24, 30. 12, 8, 6, 9, 12, 15.

$e, f, g,$ $m, n, o, p.$ $\therefore h : l = m : p,$
6, 4, 3, 8, 12, 16, 20. $4 : 10 = 8 : 20.$

D. 1	E. 8 & 5.	$\because e : f = G : H = Y : Z$;
3	E. 8 & 5.	and $f : g = K : L = Z : a$;
2	22, V.	$\therefore$ *ex æq.* $e : g = Y : a$.
4	H. 3 & 5.	And $\because$ A : B $\therefore$ $S : T = e : g$;
5	11, V.	$\therefore S : T = Y : a$;
6	Pr. B, V.	and $\therefore$ *inv.* $T : S = a : Y$.
7	E. 7 & 22, V.	But $S : X = Y : d$, $\therefore$ *ex æq.* $T : X = a : d$.
8	E. 8 & 3.	Also $\because h : k = C : D = T : V$;
9	E. 8 & 3.	and $k : l = E : F = V : X$;
10	22, V.	$\therefore$ *ex æq.* $h : l = T : X$.
11	*Sim.*	So $m : p = a : d$, and $T : X = a : d$;
12	Con. 11, V.	$\therefore h : l = m : p$;
13	Rec.	Therefore, *if there be any number of ratios, &c.* Q. E. D.

SCH, Propositions F, G, H, and K "are annexed to the 5th book," says SIMSON in his Notes," because they are frequently made use of by both ancient and modern geometers. And in many cases, compound ratios cannot be brought into demonstration, without making use of them."

And SIMSON adds, "Whoever desires to see the doctrine of ratios delivered in this 5th book solidly defended, and the argument brought against it by And. Tacquet, Alph. Borellus, and others, fully refuted, may read Dr. Barrow's mathematical lectures, viz., the 7th and 8th of the year 1666."

Fuller information, if desired, may be obtained from Geometry, Plane, Solid and Spherical, Bk. II., pp. 31—78. DE MORGAN'S *Study and Difficulties of Mathematics*,—On Proportion, ch. XVI, pp. 79—86. *Arithmetic and Algebra*, pp. 33—39. *Connection of Number and Magnitude* and *Proportion* and *Ratio*. *Penny Cyclopædia*, Vol. XIX., p. 49 & 307.

Prop. L.—Theor.

A compound Ratio is equal to the product of its component simple ratios.

Dem. **Def. A, V.** **In any number of magnitudes of the same kind, the first to the last has the ratio compounded of the ratio of the 1st to the 2nd, of the 2nd to the 3rd, of the 3rd to the 4th, and so on to the last magnitude.**

Pr. F, V. Ratios compounded of the same ratios are the same to one another.

7 A, V. The ratio of two lines is the same as that of the numbers which express the number of times that any third line is contained in them respectively.

15, V. Magnitudes have the same ratio to one another which their equimultiples have.

E. 1	Hyp.	Let the ratio A : B = the ratios compounded of C : D and of E : F;
2	Conc.	then $\frac{A}{B} = \frac{C}{D} \cdot \frac{E}{F}$.
C.	Sup.	Let ratio C : D = ratio G : H, and ratio E : F = ratio H : K.
D. 1	Def. A, V.	∵ G : K is compd. of G : H and H : K;
2	C.	and ∵ G : H and H : K = C : D and E : F;
3	H.	and ∵ also the ratio compounded of C : D and E : F = A : B;
4	Pr. F, V.	∴ the ratio G : K = ratio A : B.
5	Pr. 7 A, V. & 15, V.	But $\frac{G}{K} = \frac{g}{k} = \frac{mg}{mk} = \frac{g}{m} \cdot \frac{m}{k} = \frac{c}{d} \cdot \frac{e}{f}$;
6	D. 4.	and ratio $\frac{A}{B}$ = ratio $\frac{G}{K}$;
7	Conc.	∴ the comp. ratio $\frac{A}{B} = \frac{C}{D} \times \frac{E}{F}$.

Q. E. D.

Arith. Illustration. Let $\frac{16}{64}$ be compounded of $\frac{2}{4}$ and $\frac{8}{16}$;

$$\text{then } \frac{16}{64} = \frac{2}{4} \times \frac{8}{16}.$$

$$\text{Take } \tfrac{2}{4} = \tfrac{3}{6}, \text{ and } \tfrac{8}{16} = \tfrac{6}{12};$$

$$\because \frac{3}{12} \text{ is compounded of } \frac{3}{6} \text{ and } \frac{6}{12}, \text{ or of } \frac{2}{4} \text{ and } \frac{8}{16}$$

$$\therefore \frac{3}{12} = \frac{16}{64};$$

$$\text{But } \frac{3}{12} = \frac{1}{4} = \frac{3 \times 1}{3 \times 4} = \frac{1 \times 3}{3 \times 4} = \frac{1}{2} \times \frac{4}{8};$$

$$\text{And } \frac{16}{64} = \frac{3}{12}; \therefore \frac{16}{64} = \frac{2 \times 8}{4 \times 16}.$$

Prop. M.—Theor.

If there be two fixed magnitudes, A and B, which are the limits of two others, P and Q, (that is, to which P and Q, by increasing together, or by diminishing together, may be made to approach more nearly than by any the same given difference), and if P be to Q always in the same given ratio of C to D; then A shall be to B in the same ratio.

Con. N. B. In the *first* case, that of *commensurable* proportion, the obvious principle is assumed, that to two given magnitudes of the same kind, and a third there is some magnitude which is a fourth proportional; but in the *second* or *other* case, that of *incommensurable* proportion, we can only approximate to the fourth proportional, as we approximate to the ratio of the two magnitudes numerically; since, however, such approximation may be contained without limit, it is *presumed*, that there is some magnitude between them, which is to the given third magnitude in the same ratio which the second has to the first; that is, some magnitude which is a fourth proportional to the three.

Dem. 11, V. Ratios that are the same to the same ratio, are the same to one another; *or* Magnitudes A, B and C, D, which have the same ratio with the same magnitudes P, Q, have the same ratio with one another.

14, V. If four magnitudes of the same kind be proportionals, then if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

First. Let P and Q, *by a continual increase*, approach to A and B, respectively, so that P and Q can never equal, much less exceed, A and B, but may be made to approach A and B more nearly than by any the same difference.

C.	As above	Take a magnitude B', such that $A : B' = C : D$

Sup. If $B' \neq B'$, then B' either $< B$, or $> B$.
1°. *Let B' be $< B$, or $B' = B—b$.*

D. 1	H. & C.	$\because P : Q, = C : D$, and $A : B' = C : D$;
2	11, V.	$\therefore A : B' = P : Q$;
3	14, V.	But A always $> P$, $\therefore B'$ always $> Q$.
4	D. 3, & H.	Now $\because$ $Q < B'$ and $B' < B$ by the difference b,
5	Conc.	$\therefore$ Q cannot approach B within the difference b;
6	Remk.	,— but this is contrary to the hypothesis;
7	Conc.	$\therefore B'$ cannot be $< B$.

2°. *Let B' be $> B$; and take A' such that $A' : B = A : B'$.*

D. 1	14, V.	Then $\therefore$ $B < B'$, A' is $< A$, as by the difference a;
2	H.	And $\because$ $A' : B = A : B'$ & $P : Q = A : B$;
3	11, V.	$\therefore A' : B' = P : Q$.
4	18, V.	but B always $> Q$, $\therefore A'$ always $> P$.
5	D. 4.	Wherefore, $\because$ P is always $< A$, and $A' < A$ by a;
6	Conc.	$\therefore$ P cannot approach A within the difference a,
7	Remk.	but this is contrary to the hypothesis;
8	Conc.	$\therefore B'$ cannot be $> B$;
9	D. 7, I. & D. 8.	And $\because$ B' neither $<$ nor $>$ B;
10	Conc.	$\therefore B' = B$, *i. e.*, $A : B = C : D$.

Second. Let P and Q approach to A and B respectively by a *continual decrease.*

D. 1	*Sim.*	In the same manner, by substituting "greater" for "less," and "less" for "greater," we demonstrate,
2		that B′ also is neither $>$ nor $<$ B;
3	Conc.	again $\therefore$ B′ $\doteq$ B, *i.e.*, A : B = C : D.
4	Rec.	Therefore, *If there be two fixed magnitudes, &c.* Q. E. D.

USE & APPL. The Author of *Geometry, Plane, Sol. and Spher.*, p. 46, says of this Proposition,—"it will be found of very extensive application in Geometry. By help of it, the lengths of plane curves, and the areas bounded by them, the curved surfaces of solids, and the contents they envelope, may in many instances be brought into comparison with little greater difficulty than right lines, rectilineal areas, and solids bounded by planes." "But the use of the proposition is by no means confined to these. It may be regarded as one of the first steps to what is called the higher Geometry, and in this view likewise, is well worth the attention of the student."

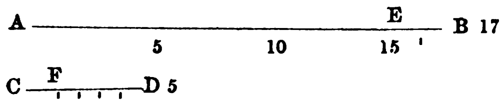
PROP. N.—PROB.

To find a common measure of two lines.

SOL. & DEM. Def. 1, V.—*Note.* One magnitude *measures* another when it is contained in that other magnitude an exact number of times.

And a magnitude which is a measure of two or more magnitudes is named the *common measure* of those Magnitudes.

E. 1	Dat.	Let A B and C D be two lines, or magnitudes;
2	Quaes.	to find a line which will be contained exactly both in A B and in C D;
S. 1	SUP. 1.	If C D is contained exactly in A B, then C D measures A B;—and then also any aliquot part of C D will be the common measure both of C D and A B; *or*
2	Def. 1, V.	$\because$ m C D = A B, $\therefore$ C D is a measure of A B;
3	"	and $\because$ m C F = C D, $\therefore$ C F a com. meas. of C D and A B,



4	Sup. 2	But if m CD = AB — EB, and n E B = C D;
5	Def. 1, V.	then n E B also measures A E, a mult. of C D,
6	Conc.	and ∴ E B will measure A B, and ∴ is a com. measure of A B and C D.
7	Sup. 3.	But if E B does not measure C D,
8	Hyp.	let 2 E B = CD — C F, *i. e.*, = D F.
9	S. 8.	Then ∵ E B measures D F, if C F measures E B, C F also will measure D F;
10	S. 8	and D F is a mult. of E B;
11	Conc.	∴ C F a measure of C D, and of A E and also of A B.
12	Sup. 4.	Let 2 C F = E B;
13		then C D = 2 E B + C F = 4 C F + C F = 5 C F;
14		and A B = 3 C D + EB = 15 C F + 2 C F = 17 C F.
15	Conc.	∴ C F is contained in C D *five* times, and in A B *seventeen* times; and C F is the *com. meas.* of C D and A B.
16	*Sim.*	Thus may be found the *com. meas.* of any other two commensurable lines.

Cor. 1. "*The greatest common measure of the remainder and lesser magnitude is also the greatest common measure of the two magnitudes,*" For

∵ every com. meas. of A and B is also a com. meas. of B and R—the remainder;

∴ the greatest com. meas. of A and R will be found among the com. measures of R and R.

Now every one of the latter measures both A and B;

∴ the greatest among them is the greatest com. meas. of A and B.

Cor. 2. *Any aliquot part or submultiple of a common measure, is also a common measure.*

Cor. 3. *By repeating the process with the remainder and lesser magnitude, and again with the new remainder (if there be one) and the preceding, and so on,* the greatest common measure *of two given commensurable magnitudes, A and B, may be found.*

S. 1	Hyp.	Let $2B = A - R$; $3R = B - R_2 = R - R_3$ and $5R_3 = R_2$ exactly.
2	Cor. 1, N, V.	Then $\because$ the greatest *com. meas.* of A & B, is also the greatest for B & R;
3	"	and the greatest for B & R, the greatest for R & R_2 and so on;
4	Hyp.	and $\because$ R_3 is contained in itself and also in R_2 exactly;
5		$\therefore$ R_3 is the greatest common measure of R_2 & R_3,—
6		and $\therefore$ also the greatest common measure of A & B.

Cor. 4. *Any two commensurable lines are to one another as the numbers denoting the number of times that they respectively contain their common measure; thus, if the com. meas. of* A B *be* E F, *taken* 5 *times; and that of* C D, *the same* E F, *taken* 7 *times, then the ratio* A B : C D = *the ratio* 5 : 7.

Sch. If, on continuing the process, we never arrive at a remainder which exactly measures the preceding remainder, the magnitudes are *incommensurable.*

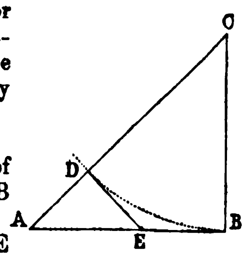
Use & Appl. The Process *for finding the greatest common measure of two numbers* is included in this Prop. N, Bk. V:—for, by dividing the greater number by the less, and finding the remainder;—then, by dividing the less by the remainder, and finding the second remainder, if there be one; and by dividing the first remainder by the second, and finding the third remainder; and so on, until a remainder be found which exactly measures the last preceding remainder; this final remainder that exactly measures the last preceding will be the greatest common measure.

PROP. O.—THEOR.

The diagonal and side of a square are incommensurable.

CON. Pst. 3, I. A circle may be drawn at any distance from the centre.
11, I. To draw a perpendicular from a given point.
DEM. 32, I. The int. ∠s of every △ are together equal to two rt. ∠s.
20, I. Any two sides of a △ are together greater than the third side.
Def. 15, I. Every point in the ⊙ce of a circle is at an equal distance from the centre.
Cor. 3, 16, III. Tangents to a ⊙ from the same · are equal.
37, III. If from a point without a circle there be drawn two lines, one of which cuts the circle and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets it shall touch the circle.
5, I. The ∠s at the base of an isosc. △ are equal.
6, I. If the ∠s at the base of a triangle, are equal the sides opposite are equal.

E.	1	Hyp.	Let A B C be the half of a square; A C the diagonal, and B A, B C two sides conterm. in B.
	2	Concl.	Then the side B A, or B C, does not measure A C; nor have B A and A C any common measure.
C.	1	Pst. 3.	From C, the extr. of A C, with rad. C B draw arc B D;
	2	11, I.	and from D draw D E ⊥ A C.
D.	1	H. & 32, I,	∵ ∠ B is a rt. ∠, and ∠s A, C each < a rt. ∠;
	2		∴ A B < A C.
	3	20, I.	Again ∵ AB+BC>AC, or 2 AB>A C;
	4	D. 3 & 2.	∴ A C > A B, but < 2 A B.
	5	*Sim.*	And ∵ the same is true of every square;



6		$\therefore$ AC — AB = AD, and AD is $<$ AB.
7	Def. 15, I & H	Also CD = CB = AB, and AD $<$ AB;
8	Cor. 16, III.	and $\because$ ED and EB are tangts. from the same $\cdot$ E;
9	37, III.	$\therefore$ E D = E B.
10	C.	And $\because$ A D E is a $\triangle$, and $\angle$ A D E a rt. $\angle$; and $\angle$ A = $\frac{1}{2}$ a rt. $\angle$;
11	5, I. & 6, I.	$\therefore$ $\angle$ DEA = $\frac{1}{2}$ a rt. $\angle$, and $\therefore$ AD = DE = EB.
12	Remk.	Now when A D the first rem., or its equal E B is taken from A B,
13		then the rem. A E is the diag. of a sq., of which A D, D E are the sides.
14	*Sim.*	The same process as before will then have to be followed out;
15		and when A D as side has been taken from A E as diag.;
16		then the rem. lines will again be side and diag.
17	D. 12—16	But $\because$ *a diag. — a side* always leaves a *remainder*.
18	Conc.	$\therefore$ in this process there will *ever* be a rem.;
19		$\therefore$ the process will never terminate;
20	Conc.	and $\therefore$ A C the diag. of a sq., and C B a side, are *incommensurable*. Q. E. D.

Prop. P.—Theor.

If four straight lines, A, B, C, D, *be proportionals, (whether commensurable, or incommensurable,) the rectangle under the extremes* A·D *will be equal to the rectangle under the means* B. C.

Book II. p. 145. The numerical area of a rectangle is obtained by supposing the two sides containing the rectangle to be divided into a number of linear units of the same kind, as inches, feet, &c., and then multiplying the units on one side by the units on the other: the product represents the area or enclosed space.

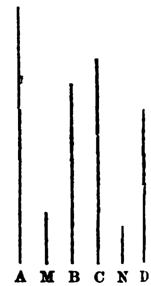
Cor. 1. Pr. 29, I. § 4. p. 18. *Geom. Plane, Sol. & Spher.* "If there be two st, lines, one of which is contained an exact number of times in one side of a rectangle, and the other an exact number of times in the side adjoining it; then, the rectangle under those two st. lines shall be contained as often in the given rectangle, as is denoted by the product of the two numbers which denote how often the lines themselves are contained in the two sides."

Def. 5, V. Criterion of the Equality of Ratios. Pr. M, Bk. V.

First. Let A & B *be commensurable, and* $\therefore$ *also* C & D.

C. 1	*Assum.*	Take any com. ratio whatever, as 7 : 5;
2		and for com. measures, M and N;
3	Pst. 2, V.	let M be contained in A *seven* times, in B *five* times; & N be contained in C *seven* times, in D *five* times.
D. 1	C. 3	$\because$ A $= 7\,m$, and D $= 5n$,
2		$\therefore$ rect. A $\cdot$ D $= 7 \times 5\,(m.n)$ $= 35\; m\,.\,n.$
3	*Sim.*	So, rect. B $\cdot$ C $= 5 \times 7\,(n.m)$ $= 35\; m\,.\,n;$
4	Ax. 1, I.	$\therefore$ rect. A $\cdot$ D $=$ rect. B $\cdot$ C

Second. Let A & B *be incommensurable, and* $\therefore$ C & D.



C. 1	Pr. M, V.	Find st. lines P & Q which approach nearer A & C than any assigned difference,
2		and let P & Q contain like parts of B & D, so that P . D = Q . B.
D. 1	Pr. M, V.	Now, $\because$ by taking like parts of B & D, continually less and less, P & Q, increase towards A & C *within* any assigned difference;
2		$\therefore$ P . D and Q . B, by increasing together. approach more nearly A . D and C . B than any assigned difference;
3	Pr. M, V.	$\therefore$ rect. A . D = rect. C . B or B . C.
4	Rec.	Therefore, *If four st. lines, &c.,* Q. E. D.

USE & APPL. The Theory of Proportion in Arithmetic and Algebra is founded on a similar truth; namely,

If four magnitudes be proportionals, and if A, B, C, D, *represent those magnitudes numerically,* i.e., *if* A *and* B *represent the numbers of times, the unit of their kind is contained in the two first, and if C and D represent the numbers of times, the unit of their kind is contained in the two last, then the quotient or fraction* $\frac{A}{B}$ *shall be equal to* $\frac{C}{D}$; *and conversely,"* See *Geom. Plane, Sol. and Spher. p.* 46 & 47.

EXAMPLES OF REASONING BY PROPORTION.

1°. If A 2 : B 4 : C 8;
then A 2 : C 8 = A^2 4 : B^2 16, by Def. 10, V.

2° If A 2 : B 4 : C 8 ; D 16;
then A 2 : D 16 = A^3 8 : B^4 64, by Def. 11, V.

3°. If A 6 : B 9 = C 8 : D 12;
invertendo, B 9 : A 6 = D 12 : C 8;

4° If A 9 : B 6 = C 12 : D 8;
alternando, A 9 : C 12 = B 6 : D 8, by 16, V.

5°. If A 5 : B 4 = C 10 : D 8;
dividendo, A 5 — B 4 : B 4 = C 10 — D 8 : D 8, by 17, V.

6° If A 5 : B 3 = C 10 : D 6;
componendo, A 5 + B 3 : B 3 = C 10 + D 6 : D 6, by 18, V.

7° If A 5 : B 4 = C 10 : D 8;
convertendo, A 5 : A 5—B 4 = C 10 : C 10—D 8. by Pr. E, V.

8°. If
$$\begin{array}{cccccccc} A\ 12 & & B\ 6 & & C\ 18 & & D\ 36 & \\ & : & & : & & : & & ; \\ E\ 6 & & F\ 3 & & G\ 9 & & H\ 18 & \end{array}$$

ex æquali, A 12 : D 36 = E 6 : H 18. by 22, V.

9°. If
$$\begin{array}{ccc} A\ 12 : B\ 6 & & C\ 3 : D\ 9 \\ & \times & ; \\ E\ 4\ : F\ 12 & & G\ 6 : H\ 3 \end{array}$$

ex. æq. perturbato, A 12 : D 9 = E 4 : H 3. by 23, V.

10°. If A 12 : B 6 = C 6 : D 3;

miscendo. A 12 + B 6 : A 12 — B 6 = C 6 + D 3 : C 6 — D 3. Use and App. 22, V.

11°. If A 5 : B 4 = C 10 : D 8,

permutando, A 5 : C 10 = B 4 : D 8, by 16, V.

invertendo, C 10 : A 5 = D 8 : B 4, by B, V.

componendo, C 10 + A 5 : A 5 = D 8 + B 4 : B 4, by 18, V.

REMARKS ON BOOK V.

To the Notes and Observations gathered from various sources we simply add the commendation of Billingsley, fol. 126.

"This fifth booke of Euclide is of very great commoditie and vse in all Geometry, and much diligence ought to be bestowed therin. It ought of all other to be throughly and most perfectly and readily knowne. For nothyng in the bookes followyng can be vnderstood without it: the knowledge of them all depende of it.

And not onely they and other writinges of Geometry, but all other Sciences and also artes : as *Musike*, *Astronomy*, *Perspective*, *Arithmetique*, the arte of accomptes and reckoning, with other such like. This booke therefore is as it were a chiefe treasure, and a peculiar iuell much to be accompted of. It entreateth of proportion and Analogie, or proportionalitie, which pertayneth not onely vnto lines, figures, and bodies in Geometry; but also vnto soundes & voyces, of which Musike entreateth, as witnesseth *Boetius* and others, which write of Musike. Also the whole arte of Astronomy teacheth to measure proportions of tymes and mouings. *Archimedes* and *Iordan*, with other, writing of waightes, affirme, that there is proportion betwene waight and waight, and also betwene place and place. Ye see therefore how large is the vse of this fift booke. Wherfore the definitions also thereof are common, although here, of *Euclide* they be accommodate and applied onely to Geometry. The first author of this booke was, as it is affirmed of many, one *Eudoxus*, who was *Platos* scholer, but it was afterwards framed and put in order by *Euclide*."

GRADATIONS IN EUCLID.

BOOK VI.

THE THEORY OF PROPORTION APPLIED, FOR COMPARING THE SIDES AND AREAS OF PLANE RECTILINEAL FIGURES.

"THIS SIXTH BOOKE is for vse and practise a most speciall booke. In it are taught the proportions of one figure to an other figure, and of their sides the one to the other, and of the sides of one to the sides of an other, likwise of the angles of one to the angles of the other. Moreover it teacheth the description of figures like to figures geuen and marueilous applications of figures to lines, euenly, or with decrease or excesse, with many other theoremes, not onely of the Proportions of right lined figures, but also of sectors of circles, with their angles. On the Theoremes and Problemes of this Booke depend for the most part the compositions of all instrumentes of measuring length, breadth, or deepenes, and also the reason of the vse of the same instrumentes, as of the Geometrical square, the Scale of the Astrolabe, the quadrant, the staffe, and

such others. The vse of which instrumentes, besides all other mechanicall instrumentes of raysing up, of mouing, and drawing huge things incredible to the ignorant, and infinite other ginnes (which likewise haue their groundes out of this Booke) are of wonderfull and vnspeakeable profite, besides the inestimable pleasure which is in them."—BILLINGSLEY, fol. 153.

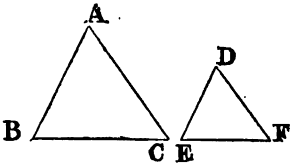
The Theory of Proportion, exhibited in the fifth book, is in the sixth applied to determining the proportions which exist between both the sides and the areas of similar plane rectilineal figures. The basis of the comparisons instituted is not identity of size, but identity of form; and when this cannot be predicated, or clearly inferred, no true Geometrical proportion can be established. The sixth book however advances further than this, and enables us to construct a figure, which shall possess the form of a first given figure and the size of a second. By the second book we may decribe a square equal to a given rectilineal figure;—by the sixth we may make any right-lined figure which we choose, equal in size to a given rectilineal figure.

We are also empowered, to find Lines and to draw rectilineal figures in proportion the one to the other; and to increase or diminish any figure according to a given Ratio. From this book we derive the principles of what is termed the Rule of Three, and the geometrical form for the solution of a quadratic equation; it extends also to a much wider application the fertile truth, that the square of the hypotenuse equals the sum of the squares of the sides of a right-angled triangle; and it supplies the easiest and most certain rules by which to conduct Measurements of all kinds. These will be seen when we show the Uses of various Propositions.

In general terms it may be said that the sixt book establishes; 1st, the proportion between the sides of similar triangles; and, 2nd, the proportion existing between the areas of similar rectilineal figures: it also lays down the methods, either of finding magnitudes proportional to other magnitudes, or of describing figures similar to other figures, or equal to them.

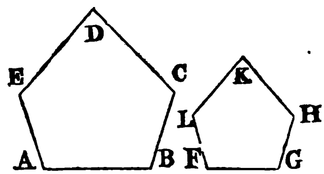
DEFINITIONS.

I. Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals: thus the $\triangle$s ABC, DEF, are similar, if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and if $AB : AC, = DE : DF$, and $AB : BC = DE : EF$.



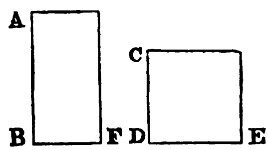
"This definition, like some others to be found in the Elements, is excessive. To contain no more than is strictly necessary (or, indeed, than as yet has appeared to be probable), it should be modified as follows:—*Two rectilineal figures are said to be similar when the first has all its sides but one proportional to the sides of the other, and* the angles included by those sides equal to the angles included by the corresponding sides of the other." *Geom. Pl., Sol. & Sph.* p. 57.

According to the definition, for one rectilineal figure to be similar to another, the conditions to be fulfilled are equal to twice the number of



sides, or rather to the sum of the number of sides and of the number of angles. Thus in the pentagons A B C D E, F G H K L; 1°. ∠ A = ∠ F; 2°. ∠ B = ∠ G; 3°. ∠ C = ∠ H; 4°. ∠ D = ∠ K; and 5°. ∠ E = ∠ L. Also 6°. E A : A B = L F : F G; 7°. A B : B C = F G : G H; 8°. B C : C D = G H : H K; 9°. C D : D E = H K : K L; and 10°. D E : E A = K L : L F. See HOSE's *Euclid*, p. 198.

II. "Reciprocal figures, viz., triangles and parallelograms, are such as have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of this other is to the remaining side of the first;" thus, A B : C D = D E : B F;—the analogy beginning in one figure and ending in the same.



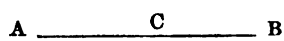
"Figures are reciprocal when the antecedents and the consequents of ratios are in each of the figures." EUCLID.

Another way of putting the definition is:—"The sides of two figures, ABF, CDE, are *reciprocally* proportional, when the *extremes* of the proportion are sides of one figure, and the *means* are sides of the other;" as AB . BF = CD . DE.

The sides are *directly* proportional, when in each figure the two sides compared are one an extreme and the other a mean: thus, if AB : BF = CD : EF, the proportion is direct.

III. A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less; thus in the line A B and its parts,

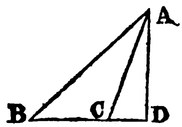
A B : A C = A C : C B.



For EUCLID's definition, LARDNER substitutes,—"when the whole line is to one segment as that segment to the remaining one."

A line thus divided is also said to be divided *medially*; and the ratio of its segments is named the *medial ratio.*—Prop. 30, VI. is the problem by which the segments are made, and is but another form of Prop. 11, II.; to divide a line so that the rectangle under the whole line and one segment shall equal the square of the other segment.

IV. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base; thus A D is the altitude of the triangle A B C.



Whichever side of a figure is assumed as the base, the altitude is the perpendicular distance from the base, or the base produced, to the point or line most distant from the base. The altitude of the same figure may vary with the change in position of its base.

SUBSIDIARY DEFINITIONS.

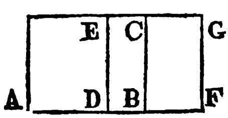
Def. A. A straight line, O C, divided into three parts, is said to be *harmonically* divided, when the whole line C O is to one of its extreme segments O E, as the other extreme C D is to the middle part D E; *i. e.*, $OC : OE = CD : DE$.

Four st. lines are said to be *harmonicals*, when they pass through the same point, and divide any one st. line harmonically.

Def. B. "A figure is given in *species*, when its several angles and the ratios of the sides about them are given."

Def. C. "A figure is given in *magnitude*, when its area, or any figure equal to it in area, is given."

Def. D. "A parallelogram is said to be applied to a straight line, when it is described upon it as one of its sides; *ex. gr.* the parallelogram A C is applied to the straight line A B.



DEF. E. " But a parallelogram A E, is said to be applied to a straight line A B, *deficient* by a parallelogram, when A D, the base of the parallelogram A E, is less than A B, the base of parallelogram A C; and because the ▱ A E is less than the ▱ A C, (described upon A B, with the same ∠ A, and between the same parallels A B, E C,) by the ▱ D C;—therefore the ▱ D C is called the *defect* of ▱ A E.

DEF. F. And a parallelogram A G is said to be applied to a straight line A B *exceeding* by a parallelogram, when A F, the base of ▱ A G is greater than the base A B of ▱ A C; and because the ▱ A G exceeds the ▱ A C, (described upon A B, with the same ∠ A, and between the same parallels A F, E G,) by the ▱ B G; therefore the ▱ B G is named the *excess* of ▱ A C.

PROPOSITIONS.

PROP. 1.—THEOR.

Triangles and parallelograms of the same altitude are one to another as their bases.

CON. Pst 1, I. A st. L. may be drawn from one · to any other point.

Pst. 2, I. A terminated st. L. may be produced in a st. L.

3, I. From the gr. of two st. Ls. to cut off a pt. = the less.

DEM. 38, I. Triangles upon equal bases and between the same parallels are equal to one another. 1, V.

Def. 5, V. The 1st of four Ms is said to have the same ratio to the 2nd which the 3rd has to the 4th, when any equims. whatsoever of the 1st and 3rd being taken, and any equims whatsoever of the 2nd and 4th; if the *m* of 1st be < = or > that of the 2nd, the *m* of the 3rd is also < = or > that of the 4th ;

41, I. If a ▱ and a △ be upon the same base and between the same ||s, the ▱ shall be double of the △.

15, V. Ms have the same ratio to one another which their equims. have.

11. V. Ratios that are the same to the same ratio, are the same to one another.

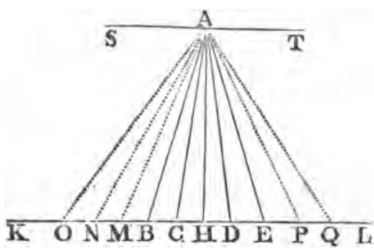
28, I. If a st. line falling upon two other st. lines makes the ext. ∠ = the int. and opp. ∠ upon the same side of the line; or makes the ∠s upon the same side together = 2 rt. ∠s; the two st. lines shall be parallel.

33, I. The st. lines which join the extrs. of two eq. and parallel st. lines towards the same parts, are also = and ∥.

36, I. ▱s on eq. bases and between the same ∥s are equal to one another.

CASE I. *Triangles of the same alt. are to one another as their bases.*

E.	1	Hyp.	Let △s ABC & ADE have the same alt. AH; *i.e.*, let BE ∥ ST;
	2	Conc.	then BC : DE = △ABC : △ADE.

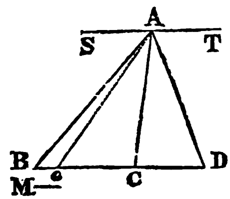


C.	1	Pst. 2, I.	Produce BE indef. to K & L;
	2	3, I.	from BK cut off, BM, MN, NO, each = B C;
	3	3, I.	and from EL cut off EP, PQ, each = DE;
	4	Pst. 1, I.	join AM, AN, AO; AP, AQ.

D.	1	C. 2 & H.	Then ∵ BC = BM = MN = NO, & KL ∥ ST;
	2	38, I.	∴ △s ABC, ABM, AMN & ANO are equal;
	3	1, V.	∴ the *m* which CO is of CB, △ACO is of △ACB;
	4	Remk.	*i. e.* OC & △ACO are equims. of BC & △ABC.
	5	*Sim.*	So, DQ & △ADQ are equims. of DE & △ADE;
	6	38, I.	and if OC > = or < DQ, △ACO > = or < △ADQ.
	7	D. 4.	Now, ∵ CO & △ACO are equims. of CB & △ACB;

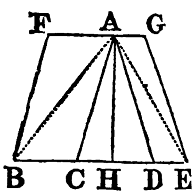
8	D. 5.	& $\because$ DQ & $\triangle$ ADQ are equims. of DE & $\triangle$ ADE;
9	D. 6.	& $\because$ $\triangle$ AOC, the *m* of $\triangle$ ACB $>=$ or $<$ $\triangle$ ADQ, the *m* of $\triangle$ ADE, as OC, the *m* of BC, is $>$ $=$ or $<$ DQ, the *m* of DE;
10	Def. 5, V.	$\therefore$ BC : DE $=$ $\triangle$ ABC : $\triangle$ ADE. Q. E. D.

Otherwise. Let B C contain the *subm.* M. 4 times, and let C D contain it 3 times; and let B*e* $=$ M; then



D. 1	38, I.	$\triangle$ ABC $=$ 4 $\triangle$ AB*e* and $\triangle$ ACD $=$ 3 $\triangle$ AB*e*;
2	15, V.	hence $\triangle$ ABC : $\triangle$ ACD $=$ 4 : 3,
3	H.	But BC : CD $=$ 4 : 3;
4	11, V.	$\therefore$ $\triangle$ ABC : $\triangle$ ACD $=$ BC : CD.

CASE II. *Parallelograms of the same alt. are also to one another as their bases.*



E. 1	Hyp.	Let $\square$s FC & GD have the same alt. AH; *i. e.*, let FAG $\parallel$ BHE;
2	Conc.	then BC : DE : $=$ $\square$ FC : $\square$ GD.
C.	Pst. 1, I.	Join AB & AE.
D. 1	C. & 41, I.	$\because$ $\square$ FC $=$ 2 $\triangle$ ABC, & $\square$ GD $=$ 2 $\triangle$ ADE;
2	15, V.	& $\because$ Ms have the same R. as their equims;
3		$\therefore$ $\triangle$ ABC : $\triangle$ ADE $=$ $\square$ FC : $\square$ GD.
4	Case 1.	But $\because$ BC : DE $=$ $\triangle$ ABC : $\triangle$ ADE;
5	D. 3.	& $\because$ $\triangle$ ABC : $\triangle$ ADE $=$ $\square$ FC : $\square$ GD;
6	11, V.	$\therefore$ BC : DE $=$ $\square$ FC : $\square$ GD.
7	Rec.	$\therefore$ *Triangles and parallelograms, &c.* Q. E. D.

COR. 1. From this it is evident that, *triangles and parallelograms that have equal altitudes are one to another as their bases; and having equal bases are as their altitudes.*

CASE I.—*they are their bases.*—See fig. 1.

C. 1	*Pon.* 1.	Place the given figures on the same st. line, as KL, or BE;
2	„ 2.	& let the triangles be on the same side of the line.
D. 1	28, I.	Then ∵ the perps. are equal and parallel;
2	33, I.	∴ the line joining the vertices will be ∥ the base.
3	*Sim.* 1, IV.	∴ as in Case I, 1, VI, the $\triangle$s having the same alts. are as their bases;
4	„	and as in Case II, 1, VI., the ▱s having the same alts. are also as their bases.

CASE II.—*they are as their altitudes.*

D. 1	38, I	∵ the $\triangle$s = rt $\angle$d $\triangle$s on eq. bases & alts.;
2	36, I.	& ∵ the ▱s = rectangles on eq. bases & alts.;
3		∴ the alt. as the base, and *vice versâ.*
4	*Sim.*	So, the dem. becomes an instance of Case I, Cor. 1.

COR. 2. *Any two triangles, or parallelograms,* T, T,′ *are to one another in the ratio compounded of the ratios of their alts., a, a′ and of their bases, b, b′.*

C.	Sup.	Take M a $\triangle$ or ▱ with alt. a and base b'
D. 1	Cor. 1, 1, VI.	∵ $T : M = b : b'$, and $M : T' = a : a'$;
2	Conc.	∴ $T : T' = \left\{ \begin{matrix} b : b', \\ a : a' \end{matrix} \right\}$, a R. compounded of the Rs of the bases and altitudes.

COR. 3. *The rectangle under two lines, A & B, is a mean proportional between their squares.*

D. 1	H.	∵ the square on A & the rect. A·B have the same alt. A;
2	1, VI.	∴ $A^2 : A \cdot B = A : B$.
3	H.	And ∵ the sq. and the rect. have the same base B;
4	1, VI.	∴ $A \cdot B : B^2 = A : B$.

COR. 4. Generally, "*if two triangles or two parallelograms be as their bases, they have equal altitudes; and if they be as their altitudes they have equal bases.*"

SCH. 1. From the principle that rectangles of the same altitude are to one another as their bases, the first Proposition might be directly inferred; for ▱s are equal to rectangles on the same base and with the same alt.; and △s are *one-half* of the Area of the respective ▱s.

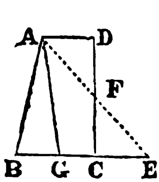
2. When, as in Case II, a first figure, FC, is to a second GD, as the base BC, or alt. AH, &c., of the first to the base DE, or alt. AH, &c., of the second,—they *vary* exactly as the base, or altitude, &c., varies. Propositions of this kind constitute a very numerous class, and are distinguished by the general name of *Variants.* Under the commercial law of supply and demand, the price of a commodity varies as these vary,—the proportion being *inverse; i. e., as* the supply increases *so* the price or estimated value diminishes. But under the mechanical law of power and work done, *as* the power increases *so* the work done increases also, the proportion being *direct.* We use the words *as* and *so* to avoid the longer and fuller enunciation; thus, the work done in a given time by a machine of one degree of power, *is* to the work done in the same time by another machine of a different degree of power, *as* the power of the first machine to the power of the second.

3. We shall do well to remember that "one quantity does not vary *as* another, because it varies *with* that other. A square and its side or root vary together, but the square does not vary *as* the side or root; for instance, if the side or root be doubled, the square is not doubled, but quadrupled;" with a side of 2 the square is 4, but with a side of 2×2, or 4, the square is 16, or four times larger. *Penny Cyclop.*, vol. xxvii, p. 137.

USE & APP. This Proposition is very frequently referred to for the demonstration of other propositions. It may also be employed *for dividing a rt. lined surface:* thus,

From a Trapezium A B C D, *with* A D ∥ B C, *to cut off a third part.*

C.		2 & 3, I.	Take CE = AD; and BG = $\frac{1}{3}$ BE, and join A G, A E;
	2	C.	then △ A B G = $\frac{1}{3}$ of trap. ABCD.
D.	1	II.	∵ AD ∥ BE, ∴ △s ADF and FCE are eq. ang.;
	2	C. 1, & 26, I.	and ∵ A D = C E, ∴ △ ADF = △ FCE;
	3	Ax. 2, I,	∴ △ A B E = trap. A B C D.
	4	C. & 1, VI.	Now △ A B G = $\frac{1}{3}$ of △ A B E;
	5	Ax. 1, I.	∴ △ A B G = $\frac{1}{3}$ of trap. A B C D.



Q. E. D.

Prop. 2.—Theor.

If a st. line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally; and conversely, *if the sides, or sides produced, be cut proportionally, the st. line which joins the points of section shall be parallel to the remaining side of the triangle.*

Dem. 37, I. △s upon the same base and between the same ‖s are equal to one another.

7, V. Eq. Ms have the same R to the same M; and conversely.

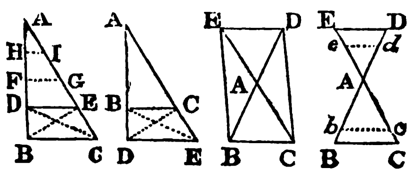
1, VI.—11, V. B, V. If 4 Ms are proportionals, they are proportionals also when taken inversely.

9, V. Ms which have the same R to the same M are eq. to one another; and those to which the same M has the same R are eq. to one another.

39, I. Eq. △s upon the same base and upon the same side of it are between the same parallels.

E.	1	Hyp. 1.	Case. I. Let DE be ‖ BC, one of the sides of △ ABC;
	2	„	and let DE cut the other sides, AB, AC, in D,E;
	3	Conc.	then BD : DA = CE : EA; or AD : DB = AE : EC.
C.		C.	Join BE & CD.

Fig. 1. 2. 3. 4.



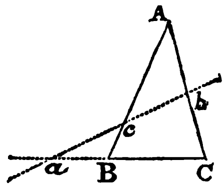
D. 1	H. & C.	∵ On a com. base DE, and between the ∥s BC, DE,
2	37, I.	∴ △BDE = △CDE.
3	Remk.	But ADE is another △ :
4	7, V.	∴ △BDE : △ADE = △CDE : △ADE.
5	C. & H.	Now, ∵ △s BDE & ADE have the same alt,;
6	1, VI.	∴ △BDE : △ADE = BD : DA.
7	*Sim.*	So, △CDE : △ADE = CE : EA;
8	11, V. B V.	∴ BD : DA = CE : EA, & *inv.* AD : DB = AE : EC.

E. 1	Hyp. 2.	CASE II. Next, in △ABC let AB, AC, or AB, AC produced, be cut in D & E;
2		and so that BD : DA = CE : EA;
3	Conc.	then DE ∥ BC.
C.	Pst. 1, I.	Make the same construction
D. 1	H. 2, 1, VI.	∵ BD : DA = CE : EA; & BD : DA = △BDE : △ADE;
2	1, VI.	& ∵ CE : EA = △CDE : △ADE;
3	11, V.	∴ △BDE : △ADE = △CDE : △ADE;
4	9, V.	∴ △BDE = △CDE;
5	C.	And these △s are on the same base:
6	39, I.	∴ DE ∥ BC.
7	Rec.	∴ *If a st. line be drawn parallel, &c.* Q.E.D.

COR. In the same manner it may be shown that, *if the sides AB, AC, of an angle be cut by any number of parallels BC, DE, FG, HI, any two parts of the one will have the same ratio to one another, as the corresponding parts of the other*, *i. e.*, the sides will be similarly divided; and every pair of corresponding segments in each side will be proportional; BH : HF = CI : IG; FD : DB = GE : EC, &c.

SCH. In the first part of the Proposition the Enunciation is not sufficiently exact, and in the other part it is "strictly speaking *false*, inasmuch as a line may cut two sides *proportionally*, and yet not be parallel to the third side." The Enunciation should be, "1°. If a line be drawn parallel to any side of a triangle, it divides the other sides, or those sides produced; so that their segments between the parallel and the third side shall have the same ratio to their segments between the parallel and the vertex of the opp. angle; and 2°. if a line cut the two sides in this manner, it will be parallel to the third side." LARDNER'S *Euclid*, p. 178.

2. The Theory of *Transversal* Lines, that is, of lines drawn across several others so as to cut them all either internally or externally, is intimately connected with this second Proposition: where, however, in the last figure, the transversal D E, cutting two of the sides A B, A C, proportionally, does not cut the third side B C, except on the idea that the point of external section is at an infinite distance, the segments being equal. In triangle ABC in the margin, the line the *a c b* is the transversal, cutting A C, A B internally in the points *b*, *c*, and C B externally in the point *a*. Here and in all similar cases, Ac × Ba × Cb = Bc × Ca × Ab.



In the Projection of figures, and in Surveying, especially, when inaccessible points are required by the aid only of signal poles and a measuring line, the theory of Transversals is very useful; for its leading principles, however, we refer to Appendix III. pp. 324—332, of LARDNER's Euclid.

3. The ways in which a st. line A B, may be cut in a given ratio, A D : D B, belong to the full consideration of Prop. 2, Bk. VI. Here one point of section will be D. Take C, the point in which A B is bisected, and make C E = CD. Thus, BE = AD and AE = B D; ∴ B E : E A is the given ratio, and E another required point of section. In a given st. line there are thus two points of internal section.

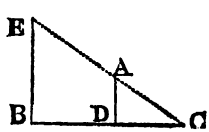
In the same way, if A F : B F & A G : B G be each the given ratio, F & G are two points of external section, cutting the line A B produced in the given ratio. There are, therefore, four points, two internal and two external, at which a line may be cut in a given ratio.

4. This Proposition shows, moreover, that parallel lines, as E D, *e d*, *b c*, BC, fig. 4, cut diverging lines A B, A C, A D, A E, proportionally.

USE & APP. 1. The second proposition, bk. VI., is of absolute necessity in establishing many of the following propositions.

2. It is used also *for the measurement of the height of an inaccessible object* B E, *which casts an accessible shadow* B D.

Take A D, a staff of known length, and setting it upright at D, the extremity of the shadow B D, measure the shadow D C; also measure the shadow B D.



Now, $\because$ AD $\parallel$ EB,
$\therefore$ CD : DB = AD : EB;
whence $BE = \frac{DB \cdot AD}{CD}$

Or the staff A D may be set along B C the shadow of B E at a point D, where the extremities of the shadows from AD and B E would coincide, as at C; then CD : CB = AD : BE;—whence $BE = \frac{CB \cdot AD}{CD.}$.

3. *A given line* A C, as in fig. 1, *may be divided into parts proportional to those in another line* A B.

C.	1	3, I.	From A draw a line, on which the parts in A B are set, *i. e.*, A H, H F, F D, D B; and join B C.
	2	31, I.	Through the $\cdot$ s D, F, H, draw parallels to B C ;
	3	Conc.	then A C is divided into parts proportional to those in A B; *i. e.* CE : EG = BD : DF, &c.
D.		Cor. 2, VI.	The proof as in 2, VI. and Cor. 2, VI.

PROP. 3.—THEOR.

If the angle of a triangle be divided into two equal angles by a st. line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another; and conversely, *if the segments of the base have the same ratio which the other sides of the triangle have to one another, the st. line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.*

Or, "If an angle of a triangle be cut into two equal parts, and the st. line cutting the angle cuts also the base, the segments of the base shall have the same ratio to the other sides of the triangle; and if the segments of the base have the same ratio to the other sides of the triangle, the st. line drawn from the vertex to the section-point cuts into two equal parts that angle of the triangle." EUCLID.

CON. 31, I. Through a given $\cdot$ to draw a st. line $\parallel$ to a given st. line.

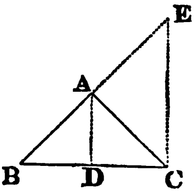
Dem. 29, I. If a st. line fall upon two st. lines, it makes the altr, ∠s equal, and the ext. ∠ = the int. and opp, ∠ upon the same side; and likewise the two int. ∠s upon the same side = two rt. ∠s.

Ax. 1, I.—6, I. If two ∠s of a △ be equal, the sides which subtend the eq. ∠s shall be equal.

2, VI.—7, V.—11, V.—9, V.—5, I. The ∠s at the base of an isosc. △ are equal; and if the eq. sides be produced, the ∠s on the other side of the base shall be equal.

E. 1	Hyp.	Case 1. Let the △ be ABC, and let AD make ∠BAD = ∠CAD;
2	Conc.	then BD : DC = BA : AC.

C.	31, I.	Through C draw CE ∥ DA, to meet BA prod. in E.
D. 1	C. 29, I.	∴ AC cuts EC ∥ AD, ∴ ∠ACE = ∠CAD;
2	H. Ax. 1, I.	but ∠CAD = ∠BAD, ∴ ∠BAD = ∠ACE.
3	C. 29, I.	Again, ∵ BE cuts the ∥s AD, EC, ∴ ∠BAD = ∠AEC;
4	D. 1. 6, I.	but ∠ACE = ∠BAD, ∴ ∠ACE, = ∠AEC, & AE = AC.
5	C. 2, VI.	And ∵ AD ∥ EC in △BCE, ∴ BD : DC = BA : AE :
6	D. 4, 7, V.	but AE = AC, ∴ BD : DC = BA : AC.



E. 1	Hyp.	Case II. Let BD : DC = BA : AC; and join AD;
2	Conc.	then in △BAC, ∠BAD = ∠CAD.

C.	31, I.	Make the same construction as in Case I.
D. 1	H. 2, VI.	∵ BD : DC = BA : AC; & BD : DC = BA : AE;
2	11, V. 9, V. 5, I.	∴ BA : AC = BA : AE; & ∴ AC=AE, & ∠AEC = ∠ACE;
3	29, I.	but ∠AEC = ∠BAD, & ∠ACE = ∠CAD;
4	Ax. 1, I.	∴ ∠BAD = ∠CAD, *i. e.*, ∠BAC is bis. by AD.
5	Rec.	∵ *If the angle of a triangle &c.* Q.E.D.

Prop. A.—Theor.

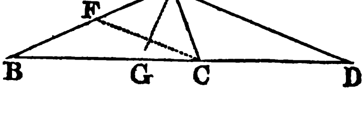
If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles, by a st. line, which also cuts the base produced; the segments between the dividing line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another; and conversely, *if the segments of the base produced have the same ratio which the other sides of the triangle have; the st. line drawn from the vertex to the point of section, divides the outward angle of the triangle into two equal angles.*

Combining Prop. III and A into one, the General Enunciation might be;

"*If the vertical angle of a triangle and its adjacent exterior angle be bisected by lines cutting the base and the base produced, the base will be cut internally and externally in the ratio of the two sides.*

Con. 31, I. Dem. 29, I. Ax. 1, I.—6, I.—2, VI.—7, V.—11, V. 9, V.—5, I.

E. 1	Hyp. 1	Case I.—Let B A a side of △ A B C be produced to E;
2	„ 2	and let A D, meeting B D the base produced, in D, bisect ext. ∠ C A E;
3	Conc.	then B D : D C = B A : A C.
C.	31, I.	Through C draw C F ∥ A D.
D. 1	C. & 29, I.	∵ A C meets the ∥s A D, F C, ∴ altr. ∠ A C F = altr. ∠ C A D;
2	H. 2, Ax. 1, I.	But ∠ C A D = ∠ D A E, ∴ also ∠ D A E = ∠ A C F.



D. 3	C. & 29, I.	Again, $\because$ BE cuts the $\parallel$s AD, FC; $\therefore$ ext. $\angle$DAE = int & opp $\angle$CFA;
4	D. 2. Ax. 1, I. 6, I.	but $\angle$ ACF = $\angle$ DAE, $\therefore$ $\angle$ACF = $\angle$ CFA, & AF = AC.
5	C. 2, VI.	And $\because$ AD $\parallel$ FC, a side of $\triangle$BCF; $\therefore$ BD : DC = BA : AF;
6	D. 4, 7, V.	but AF = AC, $\therefore$ BD : DC = BA : AC.
E. 1	Hyp.	CASE II. Let BD : DC = BA : AC, and join AD;
2	Conc.	then $\angle$CAD = $\angle$DAE.
C.	31, 1.	The same construction is to be made.
D. 1	H. & 2, VI.	$\because$ BD : DC = BA : AC, & BD : DC = BA : AF;
2	11, V. 9, V. 5, I.	$\therefore$ BA : AC = BA : AF. & $\therefore$ AC=AF, & $\angle$AFC=$\angle$ACF.
3	29, I.	But $\angle$AFC = *ext.* $\angle$EAD, & $\angle$ ACF = altr. $\angle$ CAD,
4	Ax. 1, I.	$\therefore$ also $\angle$EAD = $\angle$ CAD.
5	Rec.	$\therefore$ *If the outward angle, &c.* Q.E.D.

COR. 1. *The segments* BD, DC, *of the base produced by the external bisector* A D, *are proportional to the segments* B G, G C, *of the base* B C, *made by the internal bisector* A G.

D. 1	3, VI., A, VI.	$\because$ BG : GC = BA : AC; and BD : DC = BA : AC;
2	11, V.	$\therefore$ BG : GC = BD : DC.

"Hence, the bisectors of the internal and external angles cut the base internally and externally in the same ratio."

COR. 2. *The two lines* AG *and* AD, *which bisect the vertical angle* BAC, *and its adj. ext. angle* CAE, *cut the base produced harmonically; i. e.*, so as to make BD : DC = BG : GC;—or the base BC is a harmonic mean between its greater internal and external segments, BG and CD; *i. e.*, CD : BG = CD$\sim$BC : BC$\sim$BG.

COR. 3. *Also the two sides of a triangle* AB, AC, *and the lines* AG, AD, *which bisect the vertical and ext. vertical angles are harmonicals;* for by Def. A, VI., they are four lines passing through the same point A, and dividing a st. line, BD, harmonically.

COR. 4. *If* BG, BC *and* BD, *in the same st. line. be in harmonical progression*, DC, DG *and* DB, *will also be in harmonical progression.*

D. 1	Def. B, V.	∵ BG, BC, and BD are in harmonical progression,
2		∴ BG : BD = GC : CD.
3	16, V. B, V.	∴ *alt.* BG : GC = BD : CD, and *inv.* DC : DB = GC : BG.
4	Def. B, V.	∴ DC, DG and DB are st. lines such that the 1st DC : the 3rd DB = DC∼DG : DG∼DB.
5	Conc.	∴ DC, DG and DB are in harmonical progression.

Hence, if a st. line B C be divided in G, and produced so that D B : D C = B G : G C, the whole line made up of B G + C D, *i. e.*, B D, is divided harmonically, in the points G & C;—for D B, D G, and D C, are in harmonical progression, and therefore B D, B C, and B G are also in harmonical progression, and G and C are the points in which B D is harmonically divided.

SCH. When the given triangle is *isosceles*, the line which *bisects* the exterior vert. angle is *parallel* to the base; and the point of supposed external section of the base is then infinitely distant; or, inasmuch as two parallel lines never meet, there is in reality no point of external section. But practically, the infinite segments, of which the difference is the base, may be regarded as equal,—for the bisector is infinitely extended in both directions, and the difference between the infinite segments bears an infinitely small ratio to the segments themselves. On this theory, in the case of an isosceles triangle the proportion is preserved between the external segments and the sides. In all other cases;

E. 1	Hyp.	If the line A G, which bis. ∠ B A C cuts the base B C in G;
2	Conc.	then the st. line B D is divided harmonically in . s G, C.
D. 1	3, VI. & A, VI.	∵ B G : G C = B A : A C; and B D : D C = BA : A C;

2	11, V.	∴ BD : DC = BG : GC.
3	C.	But BG = BD − DG, and GC = GD − DC,
4	D. 2.	∴ BD : DC = BD − DG : GD − DC.
5	Def. A, VI.	∴ BD, DG and DC, are in harmonical proportion.

USE & APP. 1. *The harmonic mean, x,* between two numbers, m and n, is $\frac{2mn}{m+n}$, and *is obtained* from the harmonic proportion $m : n = m-x : x-n$;

whence $m \times (x-n) = n \times (m-x)$; *i. e.*, $mx - mn = mn - nx$;
transposing and collecting $mx + nx = 2mn$;
dividing by $m + n$, we have $x = \frac{2mn}{m+n}$ the harmonic mean.

2. By combining the Propositions 3 and A, it is proved;

1°. in *optics*,—that the axis of a pencil of rays, incident on a spherical mirror, is divided harmonically by the radiant point, the geometrical focus of the reflected rays, and the centre and surface of the reflector;

2°. in *acoustics*,—that the lengths of three musical strings, of the same thickness, material and texture, and under the same tension, must be in harmonical progression, in order to produce a note, its fifth and its octave.

PROP. 4—THEOR. (*Important.*)

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, i. e., are the antecedents or the consequents of the ratios.

OR, "Equiangular triangles have their sides proportional."

CON. 22, I. To make a △ of which the sides = three given st. lines, of which any two must be greater than the third.

32, I. The three int. ∠s of every △ are together = two rt. ∠s.

DEM. Ax. 2, I.—17, I. Any two ∠s of a △ are together < two rt. ∠s.

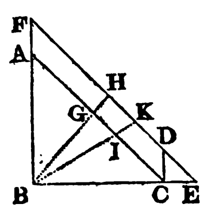
Ax. 12, I. If a st. line meets two st. lines, so as to make the two int. ∠s on the same side of it taken together < two rt. ∠s, these two st. lines being continually produced shall at length meet upon that side on which the ∠s are < two rt. ∠s.

28, I.—34, I. The opp. sides and ∠s of ▱s are = to one another, and the diam. bisects them.

2, VI.—7. V.—16, V. *alternando.* If four Ms of the same kind be proportionals, they shall also be proportionals when taken alternately.

22, V. *ex æquali.* If there be any number of Ms, and as many others, which taken two and two in order, have the same R; the first shall have to the last of the first Ms the same R which the first has to the last of the others.

E. 1	Hyp.	Let △s ABC, DCE, be eq. ang. *i.e.*, ∠ABC = ∠DCE, ∠ACB = ∠DEC, & ∴ (32, I.) ∠BAC = CDE;
2	Conc. 1.	Then these △s are sim., *i.e.*, by Def, 1, VI. the sides about each pair of eq. ∠s. are proportionals; AB : BC = DC : CE; BC : CA = CE : ED, & BA : AC = CD : DE.
3	Conc. 2.	and those sides are homologous which are opposite to the eq. ∠s.
C.	*Pon.* 22, I.	Place △DCE so that its side CE may be contiguous to BC & in the same st. line with it.
D. 1	H. *Add.*	∵ ∠BCA = ∠CED, to each add ∠ABC;
2	Ax. 2, I.	∴ ∠ABC + ∠BCA = ∠ABC + ∠CED;
3	17, I.	but ∠s ABC + BCA < 2 rt. ∠s;
4	D. 2.	∴ ∠s ABC + CED < 2 rt, ∠s;



D. 5	Ax. 12, I.	$\therefore$ BA, ED meet, if produced.
6	Sup.	Let them meet in the · F.
7	H. 28, I.	then $\because \angle ABC = DCE, \therefore BF \parallel CD$;
8	H. 28, I.	& $\therefore \angle ACB = \angle DEC, \therefore AC \parallel FE$;
9	Def. A, I.	$\therefore$ FACD is a ▱,
	34, I.	and $\therefore AF = CD$ and $AC = FD$;
10	D. 8. 2, VI.	And in $\triangle FBE$, $\because AC \parallel FE$;
		$\therefore BA : AF = BC : CE$;
11	D. 9. 7, V.	But $AF = CD$, $\therefore BA : CD = BC : CE$;
12	16, V.	& *alt.* $AB : BC = DC : CE$.
13	D. 7. 2, VI.	Again, $\because CD \parallel BF$, & $BC : CE = FD : DE$;
14	D. 9. 7, V.	but $FD = AC$, $\therefore BC : CE = AC : DE$;
15	16, V.	& *alt.* $BC : CA = CE : ED$.
16	D. 12 & 15.	And $\because AB : BC = DC : CE$,
		& $BC : CA = CE : ED$;
17	22, V.	$\therefore$ *ex æquali* $BA : AC = CD : DE$.
18	Rec.	$\therefore$ *The sides about the eq. angles, &c.* Q. E. D.

Cor. I. *If diverging lines* BF, BK, BE, *cut parallel lines* AC, FE, *the parallel lines will be cut proportionally,* i. e., $CI : IA = EK : KF$.

D. 1	H.	$\because \triangle BIC$ is similar to $\triangle BKE$;
2	4, VI.	$\therefore BI : IK = CI : EK$,
		and $BI : IK = IA : KF$;
3	11, V.	$\therefore CI : EK = IA : KF$;
4	16, V.	and $\therefore$ *alt.* $CI : IA = EK : KF$.

Cor. 2. *If two par. st. lines,* CA, EF, *be cut by any number of diverging lines,* BF, BH, BK, BE, *the parallels will be similarly cut in the* · *s of section.*

Cor. 3. *In a triangle* B F E, *a line from the vertex,* B H, *bisecting the base* E F *in* H, *also bisects the parallel to the base,* CA.

Cor. 4. *A parallel to the base of a triangle cuts off a similar triangle.*

Sch. 1. In similar figures the *homologous sides* are those which lie between equal angles as A B and C D, and also B C and CE, the $\angle$s B F E and F B E, being equal to the $\angle$s CDE and DCE;—and $\angle$s FBC, BEF, being also eq. to $\angle$s D C E and C E D. *Homologous terms* are the antecedents or consequents of a proportion.

2. **It can be said only in the case of triangles, that, if their angles are equal their sides about their equal angles are proportionals;—in rectangles and other equiangular figures of more than three sides, the corresponding sides are not necessarily proportionals; neither does it follow, as in triangles, that, if their sides are proportionals their angles are equal.**

Prop. 5.—Theor. *(Important.)*

Conversely. *If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular; and the equal angles shall be those which are opposite to the homologous sides.*

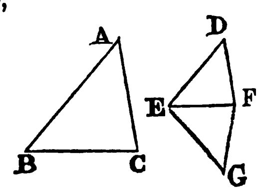
Or, *Triangles whose sides are proportional are equiangular.*

Con. 23, I. At a given · in a given line to make a rectil. ∠ = a rectil. ∠

Dem. 32, I.—4, VI.; 11, V.; 9, V.; 8, I.—Triangles having the three sides of the one equal to the three sides of the other, have the ∠s equal which are contained by eq. sides.

4, I. If two △s have each two sides and their included ∠ equal, the △s are eq. in every respect.

E. 1	Hyp. 1	Let △s ABC, DEF, have AB : BC = DE : EF.
2	,,	& BC : CA = EF : FD; and *ex. æq.* BA : AC = ED : DF;
3	Conc.	then △ A B C is eq. ang. with △ DEF, the eq. ∠s being opp. the homologous sides; *i. e.* ∠ A B C = ∠ DEF; ∠ BCA = ∠ EFD; and ∠ BAC = ∠ EDF.
C. 1	23, I.	At . s E. F, in EF make ∠ FEG = ∠ ABC, and ∠ EFG = ∠ BCA;



2	32, I.	$\therefore$ rem. $\angle$ EGF = rem. $\angle$ BAC, and $\triangle$ GEF, is eq. ang. with $\triangle$ ABC.
D. 1	32, I.	$\therefore$ $\angle$ EGF = $\angle$ BAC, and $\triangle$ GEF eq. ang. with $\triangle$ ABC;
2	4, VI.	$\therefore$ AB : BC = GE : EF;
3	H. 11, V.	but $\because$ AB : BC = DE : EF; $\therefore$ DE : EF = GE : EF;
4	9, V. and *Sim.*	$\therefore$ DE = GE; and so, DF = FG;
5	D. 4 & C.	$\therefore$ in $\triangle$s DEF, GEF, the 3 sides = the 3 sides, *i. e.* DE = GE, EF com. and DF = FG.
6	8, I. & 4, I.	$\therefore$ $\angle$ DEF = $\angle$ GEF, $\angle$ DFE = $\angle$ GFE; and $\angle$ EDF = $\angle$ EGF.
7	D, 6. C.	And $\because$ $\angle$ DEF = $\angle$ GEF, and $\angle$ GEF = $\angle$ ABC;
	Ax. 1, I.	$\therefore$ $\angle$ ABC = $\angle$ DEF;
8	*Sim.*	So $\angle$ ACB = $\angle$ DFE, and $\angle$ at A = $\angle$ at D.
9	Conc.	$\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DEF.
10	Rec.	$\therefore$ *If the sides of two Triangles, &c.* Q. E. D.

SCH. Propositions 47 and 48 of Bk. I, and 4 and 5 of Bk. VI. establish the most important properties in the Elements of Geometry, and may be regarded as universal principles in every kind of rectilineal Measurement. They are the foundation of the application of Algebra to Geometry; and, inasmuch as every rectilineal figure may be resolved into triangles, and each triangle by a perpendicular from the vertex into two rt. angled triangles, they really include the solution of all problems relating to right-lined figures.

USE & APP. I. The Theory of Representative Value, by which a small picture, executed with care to keep a due proportion between all its parts, becomes an accurate index to the human features, or of the wide spread earth itself,—*that* Theory finds its sure support on the truth, that Equiangular Triangles have their sides Proportional; and if their sides are proportional they are equiangular. It is on this Principle that Trigonometry is founded, for the sides of a Triangle have the same ratio as the sines of their opposite angles. The *Triangulation* practised in a Survey, ascertains actual distances, but the Map, founded on the measurements and calculations, is throughout constructed on the principle, that the small spaces are proportional to the real spaces; and though the lines are not drawn on the map which represent the triangle formed, for instance, by three hills not in the same st. line—yet the points themselves—

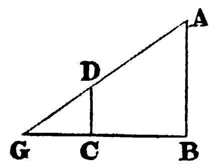
the vertices of the several angles are noted down, and their accuracy depends on the condition, that the triangle on paper is in exact ratio to the triangle in the field.

To give a general idea of the Use of Prop. 4, and of its Converse, we may state, that it is the ordinary practice in ascertaining the Lengths of inaccessible lines to construct or draw small triangles like to the large ones existing, or conceived of as existing on the ground; and as we have just stated, the principle on which this is done is derived from the last two propositions. On their truth also depends the accuracy of such Instruments as the Staff, the Quadrant, and the Geometrical Square, on which are formed triangles similar or equiangular with those which we intend to measure. Besides to take the Plane of a Place, or the outlines of a Country, from two stations, is but a continued application of the principle, that the sides of equiangular triangles are proportional.

II.—The Deductions, for Practical Purposes, made from this truth may be arranged under eight Problems and Formulas.

Prob. 1.—*Given GC & CB, the observed length of the shadows of two perpendicular objects, DC, AB, and the altitude of one, DC; to find the altitude of the other AB.*

$$\because GC : CB = DC : AB \; ; \therefore AB = \frac{CB \times DC}{GC}.$$



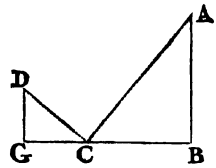
Ex.—A tower AB, casts on the level ground a shadow of 140 feet; a pole DC, standing perpendicularly, a shadow of 6 feet; the length of DC is 10 feet; required the height of the tower.

$$\text{Here, } 6 : 140 = 10 : AB \; ; \therefore AB = \frac{140 \times 10}{6} = \frac{1400}{6} = 233\tfrac{1}{3} \text{ feet.}$$

N.B.—It is said that Thales, B. C. 600, taught the Egyptians to measure the heights of the pyramids by this method.

Prob.—*By means of a mirror placed horizontally at C, the angle of incidence, ACB, being always equal to the angle of reflection, DCG:—to ascertain the height AB of a perpendicular object.*

The Observer DG, standing upright, notes the height of his eye above the horizontal line GB; the distance GC of his feet from the mirror; and also the distance CB, of the mirror from the foot of the perpendicular; two equiangular triangles are thus formed, DGC & ABC; again GC : CB = DG : AB; and

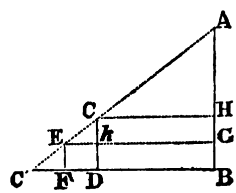


$$\therefore AB = \frac{DG \times CB}{GC}.$$

Ex. The height of the eye, DG = 5·5 ft; the distance GC = 6 feet; and CB = 96 ft; required AB.

Here $6 : 96 = 5 \cdot 5 : AB$; $\therefore AB = \dfrac{96 \times 5{\cdot}5}{6} = 88$ feet.

Prob III.—*To find the height of a perpendicular object, AB, by means of two unequal rods or poles, CD, EF, placed perpendicularly on the horizontal line CB.*



Set CD and EF, so that A may be seen in the same line with E, C, the tops of the two poles; measure EF & CD, FD and DB, *i. e.*, E k & k G;

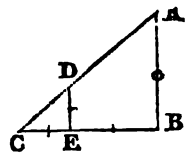
then $Ek : kG = kC : GA$; *i. e.* $GA = \dfrac{kG \times kC}{Ek}$; and $AB = GA + EF$.

Ex. By measurement EF = 5 ft; CD = 10 ft; FD = E k = 6 ft; and DB = k G = 24 ft; required the height BA.

Here $6 : 24 = (10-5) : GA$; $\therefore GA = \dfrac{24 \times 5}{6} = 20$ ft;

& 20 + 5 = 25 ft = BA.

Prob. IV. *By means of one pole DE, placed perpendicularly, to ascertain the altitude BA of a perpendicular object.*



Let DE be set upright, where A can be seen in a st. line with its top from C; measure CE, DE & EB; then; CE : EB = DE : AB;

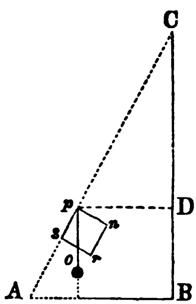
$$\text{whence } AB = \frac{EB \cdot DE}{CE}$$

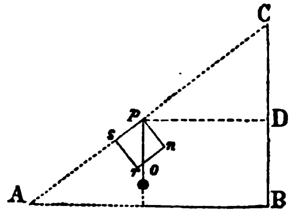
Ex. At a distance CE of 10 ft from DE = 7 ft, A can be seen in a line with point D; and the distance from E to B is 21 feet; what is the altitude BA?

$$\text{Here } 10 : 21 = 7 : BA \text{ ; } i.\ e.\ BA = \frac{21 \times 7}{10} = 14 \cdot 7 \text{ feet.}$$

Prob. V. *By means of a Geometrical Square to measure the height of an object, CB, the foot only of which we can reach.*

Let AB represent the horizontal line; *p* D a parallel to it; DB the height of the instrument above AB; CD the height of the object above *p* D; *sp* the edge of the geometrical square along which the top, C, of the object is seen; *sr*, *rn*, and *pn* graduated edges, each of 100 eq. parts, and *p* the point of suspension for the plummet.





From the place of observation measure the distance p D, and the height of the instrument, DB; direct sp towards the object C, so that s, p, and C may be points in one st. line; and note the number of parts in sr, or rn, cut off by the plummet line.

1°. When the plummet line cuts sr. in o, the triangles $p\,s\,o$ and C D p are eq. ang.; for $\because p\,o \parallel$ C B, $\angle\, s\,p\,o = \angle\, p$ C D and $\angle\, s = \angle$ D, both being rt. Δs;

$$\therefore\ s\,o : s\,p = p\,\mathrm{D} : \mathrm{C\,D};\ \text{whence CD} = \frac{sp \cdot p\mathrm{D}}{so};\ \text{and CB} = \text{CD} + \text{DB}.$$

2°. When the plummet line cuts $r\,n$ in o, the Δs, $o\,n\,p$ and C D p, are eq. ang.; for $p\,o \parallel$ C B, $\angle\, o\,p\,n = \angle$ C p D, and $\angle\, p\,o\,n = \angle\, p$ C D; $\therefore p\,n : n\,o = p$ D : D C;

$$\text{whence D C} = \frac{n\,o \cdot p\mathrm{D}}{p\,n},\ \text{and C B} = \text{C D} + \text{D B}.$$

Ex. 1. The distance p D = 80 ft.; $s\,o$ = 60 eq. pts., and D B = 6 ft.; required C B.

$$\text{Here } 60 : 100 = 80 : \text{CD};\quad \therefore \text{CD} = \frac{80 \times 100}{60} = \frac{8000}{60} = 133\tfrac{1}{3}\text{ ft.}$$

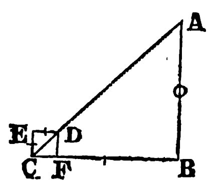
And $133\frac{1}{3}$ ft. + 6 = $139\frac{1}{3}$ = CB.

Ex. 2. The plummet line $p\,o$ cuts off $n\,o$ = 20 eq. pts.; DB = 5 ft.; and pD = 90 ft.; required CB.

$$\text{Here } 100 : 20 = 90 : \text{DC};\ \text{whence DC} = \frac{20 \times 90}{100} = 18\ \text{ft.};$$

and 18 + 5 = 23 ft. = CB.

PROB. VI. *By means of a geometrical square, with an index to point out the extremities, whether altitude or otherwise, of two objects, one of which is in the vertex of a right angle; to measure the distance.*



Place the square, each side of which is divided into 100 eq. parts, on a line, CB, which is at rt. $\angle$s to AB; and along the moveable index CD, observe the object A, and note the number of eq. parts cut off by the index on the edge of the square, DE or CE; then. $\because$ EC $\parallel$ AB, and ED $\parallel$ CB, $\therefore \angle$ E = $\angle$ B, $\angle$ ECD = $\angle$ CAB, and $\angle$ EDC = $\angle$ ACB; thus the Δs, ABC and CED are eq. ang., and $\therefore$ DE : EC = CB : BA; $\therefore$ BA = $\dfrac{\text{EC} \cdot \text{CB}}{\text{DE}}$.

Ex. 1. From the foot, B, of the perp. AB, to C, I measure BC,—it is 40 feet; the index CD, cuts off 100 eq. pts., as ED;—EC also being 100 eq. pts.; required AB.

$$\text{Here } 100 : 100 = 40 : \text{BA}; \therefore \text{BA} = \frac{100 \times 40}{100} = 40 \text{ feet.}$$

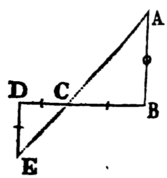
Ex. 2. The index cuts ED at 60; as before CB = 40 feet, EC or DF = 100 eq. pts.; required AB.

$$\text{Here } 60 : 100 = 40 : \text{AB}; \therefore \text{AB} = \frac{100 \times 40}{60} = 66\tfrac{2}{3} \text{ feet.}$$

PROB VII. *To find, by aid of the cross staff, or theodolite, the distance of A from B without approaching A.*

At rt. ∠s to AB, lay down the line BD; and in BD take a · C, at which place a staff; from D, but on the other side of BD, lay down DE perp. to BD, and measure along DE, until E, C & A are in one st. line; then, ∵ the ∠s at C are equal, ∠D = ∠B, and ∠E = ∠A; ∴ the △s ABC, CDE are equiangular;

$$\therefore \text{CD} : \text{CB} = \text{DE} : \text{AB}; \ i.\ e.\ \text{AB} = \frac{\text{CB} \cdot \text{DE}}{\text{CD}}.$$



Ex. The measurement of BC = 200 links; that of CD = 60 links, and of DE = 50; how many links are there between the · s B and A?

$$\text{Here } 60 : 200 = 50 : \text{BA}; \ i..\ e.,\ \text{BA} = \frac{200 \times 50}{60} = \frac{10000}{60} = 166\tfrac{2}{3} \text{ links}$$

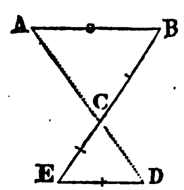
PROB. VIII.—*By means of a line, DE, of which the length is known, to find the length of its parallel, AB, one end of which, B, only can be approached.*

Set out the line BE, and on it ascertain the point C, where a line joining A and D would cross BE; measure BC, CE;

∵ the △s ACB and DCE are eq. ang.,

∴ CE : CB = DE : AB;

$$i.\ e.,\ \text{AB} = \frac{\text{CB} . \text{DE}}{\text{CE}}.$$



Ex. The parallel DE measures 800 links; the side EC, 900 links, and CB, 1800 links; required the links in AB.

$$\text{Here } 900 : 1800 = 800 : AB; \therefore AB = \frac{1800 \times 800}{900} = 1600 \text{ links.}$$

OBS. An instrument in common use,—*the Proportional Compasses*, is an example of the last Problem; in this instrument, the common centre C, about which the legs turn, is changed at pleasure, yet so as to preserve any given proportion between EC and CB, and consequently between ED and AB. These compasses give great facility in enlarging or diminishing a plan or map; and with sufficient accuracy for many purposes,—the principle being perfect, but the application liable to fail for want of the requisite care.

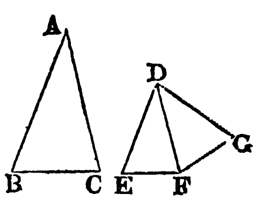
The Pentagraph and Eidograph are also instruments, of which the principle is, that the arms move parallel to each other, and consequently that the triangles formed are always similar, being exemplifications of Euclid's Prop. 4, Bk. VI. For accuracy and precision the Eidograph is far superior to the Pentagraph. See BRADLEY's *Pract. Geom.* p. 59.

PROP. 6.—THEOR.

If two triangles have one angle of the one equal to the one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

CON. 23, I.—32, I, DEM. 4, VI. 11, V. 9, V. 4, I. AX. 1, I. 32, I.

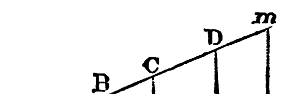
E. 1	Hyp. 1	Let $\triangle$s ABC, DEF have $\angle$ BAC = $\angle$ EDF,
2	„ 2	and BA : AC = ED : DF ;
3	Conc.	then $\triangle$s ABC, DEF are eq. ang. ; *i. e.*, $\angle$ B = $\angle$ E, and $\angle$ ACB $\angle$ DFE.



C. 1	23, I.	At ∙ s D, F in DF make $\angle FDG = \angle BAC$, or $\angle EDF$; & $\angle DFG$; $= \angle ACB$;
2	32, I.	$\therefore$ rem. $\angle B =$ rem. $\angle G$.
D. 1	C, 4, VI.	$\because \triangle DGF$ is eq. ang. with $\triangle ABC$; $\therefore BA : AC = GD : DF$;
2	H. 2.	but $BA : AC = ED : DF$;
3	11, V. 9, V.	$\therefore ED : DF = GD : DF$; & $\therefore ED = DG$.
4	C.	And $\because$ DF com., $ED = GD$, & $\angle EDF = \angle GDF$;
5	4, I.	$\therefore EF = FG$, $\triangle EDF = \triangle GDF$, $\angle DFG = \angle DFE$, & $\angle G = \angle E$;
6	C. 2. Ax. 1, I.	but $\angle DFG = \angle ACB$, $\therefore \angle ACB = DFE$;
7	H. 1. 32, I.	& $\because \angle BAC = \angle EDF$, $\therefore$ rem. $\angle B =$ rem. $\angle E$.
8	Conc.	$\therefore \triangle ABC$ is equiangular to $\triangle DEF$;
9	Rec.	$\therefore$ *If two triangles have one angle &c.* Q.E.D.

Cor. 1.—It may be added, *that the sides also about each pair of equal angles shall be proportional*; i. e., by 4, VI, $AB : BC = DE : EF$; & $BC : CA = EF : FD$.

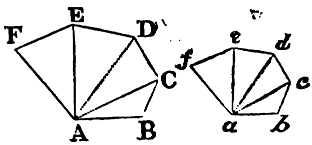
Cor. 2. *If through any points, b, c, d,* &c., *of a straight line* NM, *parallels b* B, *c* C, *d* D &c., *be drawn, which are proportional to the distances* A *b*, *b c*, *c d* &c., *from any point A on the* 1st *line, then their extremities* B, C, D &c., *will be on the rt. line* DA *passing through* A.



$\because Ab : bB = Ac : cC$; & $\angle AbB = AcC$;
$\therefore \triangle AbB$ is similar to $\triangle AcC$; & $\therefore \angle BAb = \angle CAc$;
and $\because$ A b coincides with A c; $\therefore$ AB & AC, being on the same side of NM, also coincide.

N. B.—The equation of a rt. line in Analytical Geometry depends on this principle. See LARDNER's *Euclid*, p. 184.

USE & APP. 1. Since all rectilineal figures may be divided into triangles, if any two rectilineal figures, A B C D E F *a b c d e f*, thus divided have all their angles but two equal in order, $\angle B = \angle b$, $\angle C = \angle c$, $\angle D = \angle d$, $\angle E = \angle e$, and the corresponding sides about the equal angles proportionals; then their remaining angles shall be equal each to each, $\angle A = \angle a$, $\angle F = \angle f$, and their remaining sides AF, *a f*, in the same ratio with any other two corresponding sides A B, *a b*. Thus the one figure shall be similar to the other by Def. 1, VI.



E. 1	Pst. 1, I.	Join AC, AD, AE; *ac, ad, ae.*
D. 1	H.	$\because$ $\triangle$s ABC, *a b c*, have $\angle B = \angle b$, & AB : BC $= ab : bc$;
2	6, VI.	$\therefore$ $\angle$ ACB $= \angle$ *a c b*, and AC : CB $=$ *a c : c b*;
3	H. & D. 2.	and $\because$ $\angle$ BCD $= \angle$ *b c d*, and $\angle$ACB $= \angle$ *a c b*;
4	Ax. 3, I.	$\therefore$ $\angle$ ACD $= \angle$ *a c d*.
5	D. 2. & H.	Also $\because$ AC : CB $=$ *ac : cb*; and CB : CD $=$ *c b : c d*;
6	22, V.	$\therefore$ *ex. æquo.* AC : CD $=$ *a c : c d*;
7	6, VI.	$\therefore$ $\triangle$s ACD and *a c d* are similar.
8	Sim.	And thus $\triangle$s ADE, AEF, are sim. to $\triangle$s *a d e, a e f*;
9	Conc.	$\therefore$ $\angle$ F $= \angle f$; and $\angle$s at A $= \angle$ s at *a*.
10	H.	Also $\because$ AF : AE $=$ *a f : a e*; and AE : AD $=$ *a e : a d*; and AD : AC $=$ *a d : a c*, and AC : AB $=$ *ac : a b*:
11	22, V. 16, V.	$\therefore$ *ex. æq.* AF : AB $=$ *a f : a b*; and *alt.* AF : *a f* $=$ A B : *a b*.

2. Similar rectilineal figures may also be divided into the same number of similar triangles; and their homologous or corresponding sides, are to one another in the same ratio, each to each.

PROP. 7.—THEOR.

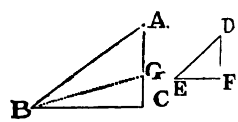
If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a rt. angle, or if one of them be a rt. angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Or, "If two triangles have one angle equal to one angle, and the sides about the other angle proportional, and the remaining angles either both less, or both not less than a right angle, the triangles will be equiangular, and will have those angles equal about which the sides are proportional."—EUCLID.

DEM. 23, I. 32, I. 4, VI. 11, V. 9, V. 5, I. 13 I. The $\angle$s which one st. line makes with another upon one side of it, are either rt. $\angle$s, or together, equal to two rt. $\angle$s

17, I. Any two $\angle$s of a $\triangle$ are together less than two rt. $\angle$s.

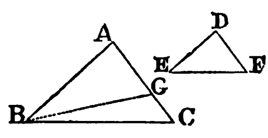
E. 1	Hyp. 1	Let $\triangle$s ABC, DEF have $\angle$ A = $\angle$ D, and AB : BC = DE : EF;
2	„ 2	and let the 3rd $\angle$s ACB, DFE, be $<$ or $\not<$ a rt. $\angle$, or one be a rt. $\angle$;
3	Conc. 1	then $\angle$s ABC, ACB, of the one = respectively $\angle$s DEF and EFD, of the other,
4	„ 2	and AB : BC = DE : EF, and BC : CA = EF : FD.



CASE I. Let each of the rem. $\angle$s C and F be $<$ a rt. $\angle$;

	Conc.	then $\triangle$s ABC and DEF, are eq. ang. $\angle$ABC = $\angle$ DEF, and $\angle$ C = $\angle$ F.
D. 1	Sup. 23, I.	If $\angle$ ABC $\neq$ $\angle$ DEF, let $\angle$ ABC > $\angle$DEF; and make $\angle$ ABG = $\angle$ DEF.
2	H. C. D. 1, 32, I	And $\because$ $\angle$ A = $\angle$ D, $\angle$ ABG = $\angle$DEF, and $\angle$ AGB = $\angle$ DFE;
3	4, VI.	$\therefore$ $\triangle$ ABG is eq. ang. to $\triangle$DEF; and $\therefore$ AB : BG = DE : EF;
4	H. 11, V.	but as DE : EF so AB : BC, $\therefore$ AB : BC = AB : B G;
5	D. 4, 9, V.	and $\because$ AB : BC = AB : BG, $\therefore$ BC = BG;
6	5, I.	and $\therefore$ $\angle$ BGC = $\angle$ BCG.
7	H. 13. I.	But $\angle$ BCG < rt. $\angle$, $\therefore$ $\angle$BGC < rt. $\angle$, and $\therefore$ adj. $\angle$ AGB > rt. $\angle$.
8	D. 2.	Now $\angle$AGB = $\angle$F, $\therefore$ $\angle$F > rt. $\angle$;
9	H.	but $\angle$F < rt. $\angle$;— which is absurd.
10	Conc.	$\therefore$ $\angle$ ABC not $\neq$ DEF, *i. e.* $\angle$ABC = $\angle$DEF, and $\angle$ A = $\angle$ D;
11	32, I.	$\therefore$ rem. $\angle$ C = rem. $\angle$ F.
12	Conc.	$\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DEF.

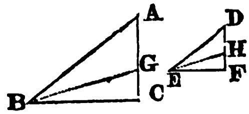
Case II. Let $\angle$s C & F be each $\nless$ a rt. $\angle$;



	Conc.	then $\triangle$ ABC, is eq. ang. to $\triangle$ DEF.
D. 1	D. 1—5 Case I.	As before BC = BG, $\therefore$ $\angle$ C = $\angle$BGC;
2	H.	but $\angle$ C $\nless$ rt. $\angle$, $\therefore$ $\angle$ BGC $\nless$ rt. $\angle$;
3	17, I.	$\therefore$ two $\angle$s of $\triangle$ BGC, are $\nless$ 2rt. $\angle$s; which is impossible;
4	Conc.	and $\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DEF, as in Case I.

CASE III. Let one of the ∠s, C, F, namely ∠C, be a rt. ∠;

	Conc.	Then also △ABC is eq. ang. to △DEF.
D. 1	Sup. 23, I.	If not, at B in AB, make ∠ABG, = ∠DEF;
2	Case 1. 5, I.	then BG = BC, and ∴ ∠BCG = ∠BGC;
3	H. Ax. 1, I.	but ∠BCG is a rt. ∠, ∴ ∠BGC is a rt. ∠;
4	17, I.	∴ two ∠s of △ BGC are $\nless$ 2 rt. ∠s; which is an impossibility;
5	Conc.	∴ △ ABC is eq. to ang. △ DEF.
6	Rec.	∴ *If two triangles have one angle. &c.*



Q. E. D.

SCH. When two angles are both greater, or both less than right angles they are both said to be *of the same affection*; and in ennunciating this Prop, instead of "both greater or both less than right angles," it is not unusual to say, "both of the same affection."

USE AND APP.—In Book I. Propositions 4, 8 and 26 contain the *criteria* of the *equality* of two triangles; and in Book VI. Propositions 4, 5, 6 and 7 may be classed together as giving the conditions on which we declare the *similarity* of two triangles. Equality in triangles is absolute, not in area only;—but the similarity is likeness of *shape*, not identity of *size*.

The *criteria* of similarity are;

1°,—The equality of the three angles, 4, VI.;

2°.—The identity of the ratios of the respective sides, 5, VI.;

3°.—The equality of two angles, one in each triangle,—and the identity of the ratios of the containing sides, 6, VI.;

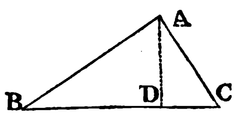
4°.—The identity of the ratios of two sides in each triangle,—the equality of an angle in each opposite one pair of homologous sides;—and each of the remaining angles opposite the other pair of homologous sides less than a right angle, or one of them a right angle.

Universally, if triangles fulfil any one of these four conditions of similarity, the sides about the equal angles are proportional.

Prop. 8.—Theor. (*Important.*)

In a rt. angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle and to one another.

Dem. Ax. 11, I. All rt. ∠s are equal to one another. 32, I. 4, VI. Def. 1, VI.



E.	1	Hyp. I.	Let △ABC have BAC a rt. ∠;
	2	”	& from A let AD be ⊥ BC the hypotenuse;
	3	Conc.	then △s ABD, ADC, are sim. to △ABC, and to each other.
D.	1	Ax. 11, 1. 32, I.	∵ ∠BAC = ∠ADB, & ∠B com.; ∴ rem, ∠ACB = rem. ∠BAD
	2	4, VI.	∴ △ABC is eq. ang. to △ABD, and the sides propls.;
	3	Def. I, VI.	∴ △ABC is similar to △ABD.
	4	*Sim.*	So, △ADC is eq. ang. and sim. to △ABC;
	5	D. 2 & 4.	and ∴ △ABD is eq. ang. & sim. to △ACD, which is eq. ang. and sim. to △ABC.
	6	Rec.	Therefore, *in a rt. angled triangle, &c.* Q. E. D.

COR. 1. *The perpendicular*, A D, *from the vertex of the rt.* ∠ BAC *to the opposite side* BC, *is* a mean proportional *between the segments* BD, DC *of this side; and also each of the sides*, BA, AC, *including the rt.* ∠ is a mean proportional *between the opposite side* BC, *and the segment of it*, BD, *or* DC *adjacent to that side*, BA *or* AC.

D. 1	H.	∵ △ ADB is sim. to △ ADC,
	Def. 1, VI.	∴ BD : DA = DA : DC;
	Obs. Def. 10, V.	*i. e.* DA is a mean proportional between BD and DC.
2	H. 4, VI.	Also, ∵ △ ABC is sim. to △ DBA,
		∴ BC : BA = BA : BD;
		i. e. A B is a mean proportional between BC and BD.
3	H. 4, VI.	And ∵ △ ABC is sim. to △ ACD,
		∴ BC : CA = CA : CD;
		i. e. AC is a mean proportional between BC and CD.

COR. 2. *The segments* BD, CD, *of the hypotenuse, made by the perp.* AD, *are to one another as the squares on the sides of the rt.* ∠, $BA^2 : AC^2$.

∵ by D. 2 & 3 of Cor. 1. 8, VI. $BD.BC : CD.BC = BA^2 : AC^2$;
if we divide the 1st & 2nd terms of the analogy by BC,
then $BD : CD = BA^2 : CA^2$

COR. 3. *The squares on the sides about the rt.* ∠ *and on the hypotenuse are to each other as the segments of the hypotenuse, made by the perp.* AD, *and the hypotenuse itself.*

∵ $BD.BC : CD.BC : BC.BC = AB^2 : CA^2 : BC^2$;
∴ on dividing the three terms by BC,
$BD : CD : BC = AB^2 : CA^2 : BC^2$.

N. B.—Several other *subsidiary* Corollaries might be added,—but the most important deductions have been given, and we subjoin only;

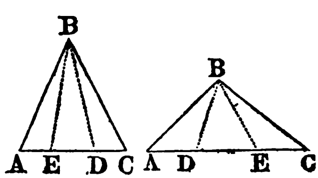
COR. 4. *If the base of a triangle*, BC, *the two sides*, AB, AC, *and the perpendicular* AD, *be four proportionals, the triangle must be right angled.*

D. 1	Hyp.	∵ in △ ABC, and in one of the component △s ABD, two sides are proportionals;
2	”	and the ∠s opp. one pair of homologous sides are equal;
3	”	and also ∵ of the ∠s opp. the other pair of homol. sides one is a rt. ∠;
4	7, VI.	∴ the whole △ ABC is sim. to the component △ ABD;
	Conc.	and ∴ also △ ABC is rt. angled.

SCH. 1. The 8th Proposition, and the deductions that may be made from it are particular cases of a more general principle; namely,

If from the vertex, B, *of a* △ ABC, *two lines* BD, BE. *be drawn to the base*, AC, *making the angles at the base*, BDA. *and* BEC, *or their supplements each equal to the vertical angle* ABC; *then the* △*s* BDA, BEC, *formed by those lines, and by the segment* DA *or* EC *which each cuts off, shall be similar to the whole triangle and to one another.*

E. 1	Hyp.	Let ABC be a △, and from vertex B, let there be drawn BD, BE, making ∠ BDA = ∠ BEC = ∠ ABC;
2	Conc.	then △ BED is isosc.; and △BDA is eq. ang. to △BEC which is eq. ang. to △ ABC.
D. 1	Remk.	When ∠ ABC > rt. ∠, ∠s BDA and BEC are ext. ∠s. at the base of the isosc. △ BDE;



2	Remk. 2.	but when $\angle$ ABC $<$ rt. $\angle$, then those $\angle$s are internal.
3	„ 3.	As the obtuse $\angle$ ABC decreases, the base DE diminishes and the sides BD, BE approach;
4	„ 4.	when $\angle$ ABC becomes a rt. $\angle$, BD and BE coincide;
5	„ 5.	and after $\angle$ ABC becomes $<$ a rt. $\angle$, BD and BE change sides.
6	„ 6.	In the general proposition, this isosc. $\triangle$ DBE, is what the perpendicular is, when the given $\triangle$ is rt. angled.
7	Conc.	$\therefore$ the sides of this $\triangle$ DBE, and the triangles under them, and the sides of the given $\triangle$ ABC, possess many of the properties already proved in the case of a rt. $\angle$d $\triangle$; for instance, as in Cor. 1, Pr. 8, VI.;

1. AC : AB = AB : AD, where AB is a mean proportional.
2. AC : CB = CB : CE, where CB is a mean proportional.
3. AC : AB = BC : BE,
4. AD : BD = BD : CE, where BD is a mean proportional.

   or since BD = BE; AD : BE = BE : CE, where BE is a mean propl.

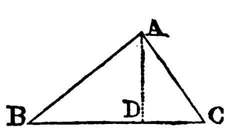
5. $\because$ AD : BD = AB : BC, $\therefore$ the segments AD and EC are in the duplicate ratio of the sides AB and BC.

2. "Hence in a right-angled triangle the segments of the hypotenuse by the perpendicular, are in the duplicate ratio of the sides." LARDNER'S *Euclid*, pp. 187, 188.

USE & APP. 1. The first Corollary of Pr. 8, bk. VI. supplies the principle on which a very clear and brief demonstration may be given of Prop. 47, bk. I.

$\because$ BC $\times$ BD $=$ BA^2,
and BC $\times$ CD $=$ CA^2;

$\therefore$ *Add.* BC . BD $+$ BC . CD
$=$ BA^2 $+$ CA^2.



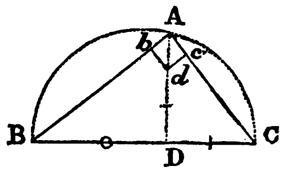
Or, BC $\times$ (BD $+$ CD) $=$ BA^2 $+$ CA^2;

i. e. BC $\times$ BC, or BC^2 $=$ BA^2 $+$ CA^2.

2. Also according to this Proposition, and by aid of a square, *inaccessible distances*, as DB, *may be measured.*

At D raise the perp. DA, and measure it;

And at A place a square, so that by looking along one of its sides A*b*, the point B may be seen in the same st. line with A*b*;



And along the other side A*c*, the point C may be seen; and measure DC;

$$\text{Then, } \because \text{CD} : \text{DA} = \text{DA} : \text{DB}; \ \therefore \text{DB} = \frac{\text{DA}^2}{\text{CD}}.$$

$$\text{Ex. Suppose AD} = 3, \text{ and CD} = 2{\cdot}25; \text{ then DB} = \frac{9}{2.25} = 4.$$

3. *In a circle any chord, as* BA, *is a mean proportional between the diameter* BC, *and that segment of the diameter* BD, *which is drawn from one extremity of the chord,* B, *and cut off by a perpendicular,* AD, *let fall from* A *the other extremity of the chord.*

D. 1	31, III. & C.	$\because$ $\triangle$s BAC and ADB have each a rt. $\angle$, and $\angle$ B common;
2	4, VI.	$\therefore$ the $\triangle$ BAC is eq. ang. to $\triangle$ ADB, and $\therefore$ BC : BA = BA : BD; *i. e.* the chord BA is a mean propl. to BC & BD.
3	8, VI.	And $\because$ $\triangle$s BAC, BDA and ADC are similar, the segments of the hypotenuse are in the duplicate ratio of the sides.

Prop. 9.—Prob.

From a given st. line to cut off any part required, i. e., any measure or submultiple.

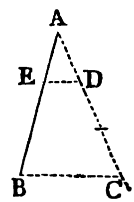
Con.—3, I. 31, I.

Dem.—2, VI. 18, V. *Componendo.* If Ms, taken separately, be propls, they shall also be propls when taken jointly ; *i. e*, if first : 2nd = 3rd : 4th ; then 1st + 2nd : 2nd = 3rd + 4th : 4th.

D, V—If 1st : 2nd = 3rd : 4th, if 1st a *m* or *pt* of the 2nd, the 3rd is the same *m* or *pt* of the 4th.

Def. 1, V.—A less M is a *pt* of a greater when the less measures the greater.

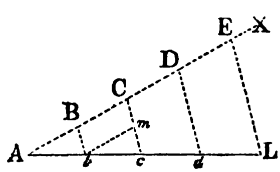
E.	1	Dat.	Let AB be a given st. line ;
	2	Quæs.	to cut off from it any *pt*. required.
C.	1	Pst. 1, I.	From. A draw AC making any ∠ with AB ;
	2	3, I.	in AC take any . D, and make AC the same *m* of AD that AB is of AE the *pt* to be cut off.
	3	31, I.	join BC, and draw ED ∥ BC ;
	4	Sol.	then AE is the submultiple required ;
D.	1	C. 3.	∵ ED ∥ BC one of the sides of △ ABC ;
	2	2, VI.	∴ CD : DA = BE : EA ;
	3	18, V.	and *compon.*, CA : AD = BA : AE ;
	4	C. D, V.	but CA is a *m* of AD ; ∴ BA the same *m* of AE ;
	5	Def. 1, V.	∴ AE the same *pt* of AB, that AD is of AC.
	6	Recap.	∵ *From a given st. line, &c.* Q. E. F.



Sch. Prop. 10, Bk. I. by which a rt. line may be bisected, and its bisections also bisected, is a particular case of this Problem.

Use & App. A simple extension of the Problem enables us; 1st, *to divide a given line, AL, into any number of equal parts.*

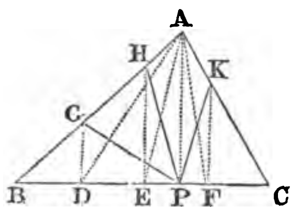
E.	1	Dat. & Quæs.	Let it be required to divide AL, into four eq. pts.
C.	1	Pst. 1, I.	Draw AX, making any ∠ with AL;
	2	3, I.	from A on AX set off four eq. spaces, *i. e.* AB = BC = CD = DE ;



3	31, I.	Join EL, and through D, C, B draw ||s to EL and cutting AL;
4	Sol.	then AL is divided into four eq. pts. in *b*, *c*, *d*, L.
D. 1	9, VI.	∵ A*b* is the same *pt.* of AL that AB is of A E; *i. e.* the 4th part;
2	*Sim.*	and ∵ A*b*, = *bc* = *cd* = *d* L;
3	Conc.	∴ AL is divided into four eq. parts.

2. *To divide a triangle,* ABC, *into any number of eq. parts,* say four, *by lines from a given point,* P, *in one of the sides, as* BC.

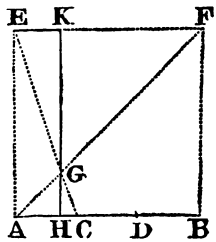
C. 1	Sch. 1. 9, VI.	Divide BC into four eq. pts. in D, E, F, and join AP;
2	31, I.	and through D, E, F draw DG, EH, FK each || AP;
3	Pst. 1, I.	then join AD, AE, AF, and PG, PH, PK;
4	Sol.	and the △s BPG, GPH, HPK, KPC, each = ¼ △ABC; *i. e.* by lines from P, △ ABC is divided into 4 eq. pts



D. 1	37, I.	∵ △ DPG = △ GAD, both being on GD and between the same ||s GD, AP;
2	*Add.* Ax. 2, I.	to each add △ BGD, ∴ △ BPG = △ ADB.
3	C 1.	Next, ∵ BD = DE = EF = FC and the altitude com.
4	38, I.	∴ △s ABD, ADE, AEF, AFC, are equal; and each is ¼ △ ABC;
5	D. 3	but △ BPG = △ ADB, ∴ BPG = ¼ △ ABC.
6	*Sim.*	So △ GPA = △ DPA, and △ HPA = △ APE,
7	*Sub.*	take △ HPA from △ GPA, and △ APE from △ DPA,
8	Ax. 3, I. D. 4.	∴ rem. △ GPH = rem. △ AED = ¼ △ ABC.
9	C.	Now fig. PHAK = APH + APK; △ APH = △ APE and △ APK = △ APF;
10	Ax. 1, I.	∴ fig. PHAK = △ APE + △ APF = ¼ △ ABC.
11	D. 5, 8, 10,	Now fig. ABPK = ¾ of △ ABC;
12	Ax. 3, I.	∴ rem. fig KPC also = ¼ of △ ABC;
13	Conc.	∴ △ ABC is divided into 4 eq. pts. by lines from · P.

3. *Given the nth part of a line AB, to find the* $(n + 1)$ *th part.*

E. 1	Dat.	Given $AC = \frac{AB}{n}$, or $AB = n \, . \, AC$;
2	Quaes.	to find $\frac{AB}{n+1}$.
C. 1	46, I. Pst. 1, I.	On AB desc. a sq. ABEF, and join AF, EC cutting in G;



2	31, I.	through G draw HGK $\parallel$ AE, cutting AB, EF, in H & K;
3	Sol.	then AH, or $EK = \frac{AB}{n+1}$; or $(n+1)\,AH = AB$.
D. 1	C.	$\because$ $\triangle$s AHG, FKG, are eq. ang.; and $\triangle$s AGC & EGF;
2	4, VI.	$\therefore$ AH : FK, or BH, = AG : GF = AC : EF, or AB.
3	Conc.	and $\therefore$ $BH = n. \, AH$; and $AB = (n+1)\,AH$;
		$\therefore AH = \frac{AB}{n+1}$. Q. E. F.

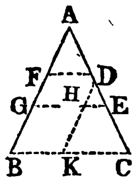
Prop. 10.—Prob.

To divide a given st. line similarly, i. e., proportionally, *to a given divided st. line;* or *to divide a given st. line into parts that shall have the same ratios to one another which the parts of the divided given st. line have.*

Or, "To divide a given undivided line similarly to a given divided line." Euclid.

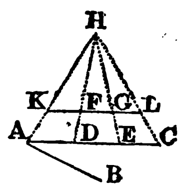
Con. Pst. 1, I. 31, I. 1, I.

DEM. 34, 1. The opp. sides and $\angle$s of ▱s are eq. to one another, and the diam. bis. them. 2, VI. 7, V. 6, VI. 28, I.



E. 1	Dat. 1.	Given AB to be divided;
2	„ 2.	& AC a line divided at the $\cdot$ s D, E;
3	Quæs.	to divide AB similarly to AC.
C. 1	*Pon.* Pst. 1, I.	Let AB and AC form an $\angle$ at A; & join BC;
2	31, I.	through D,E, draw DF,EG, each $\parallel$ BC; and also through D, DHK $\parallel$ AB;
3	Sol.	then AB is div. in F,G, as AC is in D,E.
D. 1	C. 2. 34, I.	$\because$ FH, HB are ▱s; $\therefore$ DH = FG, and HK = GB;
2	C. 2.	and $\because$ HE $\parallel$ KC, a side of $\triangle$DKC;
3	2, VI.	$\therefore$ CE : ED = KH : HD;
4	C. 7, V.	but KH = BG, & HD = GF; $\therefore$ CE : ED = BG :GF.
5	D. 2. 2, VI.	Again, $\because$ FD $\parallel$ GE in $\triangle$AGE,; $\therefore$ ED : DA = GF : FA;
6	D. 4 & 5.	and $\because$ CE : ED = BG : GF, and ED : DA = GF : FA;
7	D. 4 & 4.	$\therefore$ AB is divided in the same proportion as AC. Q. E. F.

Or,



C. 1	1, I.	On AC desc. an eq. lat. $\triangle$AHC;
2	3, I.	and from HA, HC, cut HK, HL, each = AB; & join KL;
3	Pst. 1, I.	from H draw lines to D,E, the divisions of AC;
4	Sol.	then KL = AB is div. in F,G, similarly to AC.
D. 1	C. 2. 6, IV.	$\because$ HK = HL, & HA = AC, $\therefore$ $\triangle$HKL is eq. an. to $\triangle$HAC;

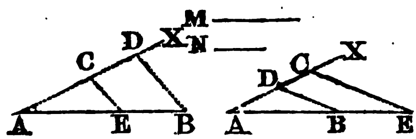
2	C. 1.	but AC = HA, ∴ KL = KH = AB.
3	*Sim.* 28, I.	So ∠HKL = ∠HAC; ∴ KL ∥ AC;
4	Conc.	and ∵ KL = AB
		∴ AB is div. similarly to AC.

Q. E. F.

Use and App.—By aid of this proposition several useful Problems may be solved.

Prob. 1. *To divide a given st. line,* AB, *internally, or externally, in a given ratio, as of* M : N.

C.	1	Pst. 1, I.	From A draw AX at any ∠ with AB;
	2	3, I.	take AC = M, and CD = N; and join DB;
	3	31, I.	Through C draw CE ∥ DB, and meeting AB, or AB produced,
	4	Sol.	Then AB is divided in the ratio M : N.

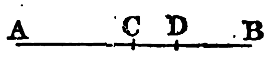


D.	1	C. 3.	∵ in △s AEC and ABD, CE ∥ DB;
	2	2, VI.	∴ AE : EB = AC : CD = M : N;
	3	Conc.	∴ AB is divided in E in the given ratio.

N. B. A line cannot be cut externally in a ratio of equality.

Prob. 2. *To find a harmonical mean between two given st. lines,* AB *and* AC.

C.	1	*Pon.*	Place the lines so as to form one st. line, AC being set off on AB.
	2	Prob. 1.	divide BC in D in the ratio AB : AC;
	3	Sol.	then AD is the harmonical mean.



D.	1	2, VI.	∵ AB : AC = BD : DC;
		Def. A. VI.	∴ AB, AD and AC are in harm. progression;
	2	Concl.	∴ AD is a harmonical mean between AB and AC.

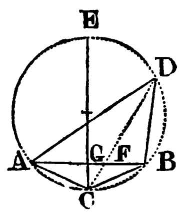
Prob. 3. *To find a third harmonical progressional to two given st. lines,* AB *and* AC.

C. 1	*Pon.*	Set off A B on AC, and produce AC;
2	Prob. 1.	divide AC produced, in D, in the ratio AB : BC;
3	Sol.	then AD is the harm. progressional required.

D. 1	2, VI. & 16, V.	∵ AD : CD = AB : BC; and *alt.* AD : AB = CD : BC,
2	Def. A, VI.	∴ AB, AC and AD, are in harm. progression,
3	Concl.	and AD is the third harm. progress. to AB, AC.

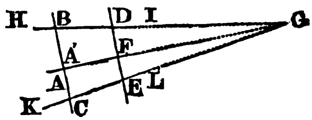
Prob. 4.—*To construct a triangle of which one side AB, the angle ADB opposite to it, and the ratio of the other sides are given.*

C. 1	33, III.	On AB desc. a ⊙ in which the ∠ = the given ∠;
2	1, III.	draw a diam. EC, at rt. ∠s to AB:
3	10, VI.	divide AB so that AF : FB = ratio of the sides;
4	Pst. 1. 2, I.	join CF, and produce CF to D in the ⊙ce;
5	Sol.	join AD, DB; and ADB is the △ required.
D. 1	C. 2. 3, III. 4, I.	∵ ∠s at G rt· ∠s; ∴ AG = GB, and chord AC = CB.
2	28, III., 27, III.	Now arc AC = arc CB; ∴ ∠ ADC = ∠CDB, and CD bisects ∠ADB;
3	3, VI,	∴ AD : DB = AF : FB, which is the given ratio;
4	Conc.	∴ △ ADB is on AB, its vert. ∠ = a given ∠, and its sides, DA : DB = a given ratio.



Prob. 5. *Through a given* · *A, to draw a line, which, on being produced, would pass through the points of intersection of two given lines,* HI, KL, *without their being produced to meet.*

C. 1	Pst. 1, I.	Through A draw any line BC, cutting HI in B, & KL in C;
2	31, I. 10, V.	draw DE ∥ BC, and divide DE so that EF : FD = CA : AB;
3	Sol.	then AF is the line required.



D.	1	Sup.	For should HI, KL, have met in G, and GF cut BC in A′;
	2	C.	then ∵ DF : FE = BA′ : A′C; ∴ BA′ : A′C = BA : AC;
	3	18, V.	and *comp.* BC : A′C = BC : AC; ∴ A′ coincides with A;
	4	Conc.	and AF is the line required.

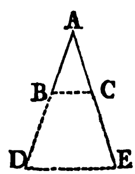
N.B. This operation is frequently called for; and the *Centrolinead* is an instrument, invented by Mr. P. Nicholson, senior, for the purpose of drawing lines tending to the inaccessible point where two given lines, if produced, would meet. Another instrument for the same purpose is the invention of Mr. John Fary. BRADLEY'S *Pract. Geom.* p. 42. There is, however, a simpler instrument used by draughtsmen, consisting of three rulers stiffly moveable about a common joint; but it is less convenient in form, and less accurate in its results, though depending on the same principle.

PROP. 11.—PROB.

To find a third proportional to two given st. lines.

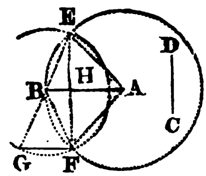
CON. Pst. 2, I. 31, I. DEM. 2, VI, 7, V.

E.	1	Data.	Given the two lines AB and AC;
	2	Quæs.	to find a third proportional to them.
C.	1	*Pon.* Pst. 2, I.	Place AB and AC to form an ∠ BAC, and produce AB and AC;
	2	3, I. 31, I.	take BD = AC, join BC, and draw DE ∥ BC;
	3	Sol.	then CE is a third proportional; *i. e.*, AB : AC : CE.



D. 1	C. 2, 2, VI.	∵ BC ∥ DE, a side of △ ADE; ∴ AB : BD = AC : CE;
2	C. 2, 7, V.	but BD = AC; ∴ AB : AC = AC : CE.
3	Rec.	∵ to the given lines AB, AC, a 3rd prop. CE, has been found. Q. E. F.

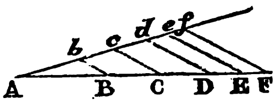
SCH. 1. There are various constructions by means of which this Problem may be solved: we select one which in practice requires the use of the compasses alone;—the given lines being AB and CD.



C. 1	Pst. 3, I.	From A with AB desc. ⊙ BEF; and from B with DC, ⊙EFG;
2		from E on arc EFG, set off CD three times to G;
3	Pst. 1, I.	join EF, AF, BF, BG, and GF;
4	Sol.	the chord FG is the third proportional.
D. 1	Con.	∵ AE, EB = AF, FB, and AB com.;
2	8, I. 5, I. 4, I.	∴ ∠ EBA = ∠ FBA; also ∠ BEF = ∠ BFE, and ∠s at H rt. ∠s;
3	D. 2. 31, III.	and ∵ ∠s at H are rt. ∠s, and ∠ EFG is a rt. ∠;
4	28, I.	∴ AB ∥ FG.
5	C. 1. 29, I.	and ∵ △s ABE, BFG, are isosc.; and ∠BFG = ∠ ABF; ∴ △ ABF is eq. ang. to △ BFG;
6	4, VI.	∴ AB : BF = BF : FG;
7	C. 7, V.	but BF = DC, ∴ AB : CD = CD : FG;
8	Conc.	∴ FG is the third proportional.

USE and APP.—1. By repeating the same construction we solve the Problem, *to continue a series of ratios in progression,* AB : BC *being the given antecedent and consequent.*

C. 1	3, I. 31, I.	Take A b = BC, and draw C c ∥ B b;
2	„ „	„ CD = bc „ D d ∥ C c;
3	„ „	„ DE = cd „ E e ∥ D d, &c.
D.	2, VI. 7, V.	It is evident AB : BC : CD : DE, EF, &c.



Arithmetically. Since $CD = \dfrac{BC \cdot BC}{AB}$, the third proportional to two given numbers is found *by dividing the square of the consequent by the antecedent*; thus, the 3rd proportional of $9 : 6 = \dfrac{6 \times 6}{9} = 4$; *i.e.*, $9 : 6 = 6 : 4$.

2. From LARDNER'S Notes on Prop. 11, Bk. VI. we take several most USEFUL THEOREMS allied to the last Problem.

THEOR. I. "*If a series of magnitudes* A, B, C, D, *be in continued proportion, their successive differences, a, b, c, d, are also in continued proportion, and in the same ratio.*

D. 1	H.	$\because A : B : C$;	*Arith.*	$2 : 6 : 18$;
2	E. V.	$\therefore$ *conv.* $A : a = B : b$.		$2 : 4 = 6 : 12$.
3	16, V.	and *alt.* $A : B = a : b$.		$2 : 6 = 4 : 12$.
4	*Sim.*	So, $\because B : C = b : c$;	So,	$6 : 18 = 12 : 36$;
5	D. 3, 4.	$\therefore a : b : c$.		$4 : 12 : 36$.
6	*Proced.*	And $a : b : c : d$, &c.	and	$4 : 12 : 36 : 108$ &c.

THEOR. II.—*If a series in continued proportion, $A : B : C : D$ &c., be an increasing one, there is no limit to the increase of its terms.*

E.	Hyp.	Let $a = B - A$; $b = C - B$; $c = D - C$ &c.; and L be the last term.
D. 1	†	$\because$ no magn. so great that we cannot obtain a greater,
2		$\therefore$ let M. be a magn. however great we please.
3		Find what multiple of a, M is;
4		and continue the series to a gr. number of terms than $\frac{M}{a}$;
5		$\therefore L > A$ by $a + b + c + d$ &c. $+\left(\text{the no. of terms} = \frac{M}{a}\right)$
6	Hyp.	But $\because a : b : c : d$ &c. is an increasing series;
7		$\therefore$ each successive term $> a$;
8		$\therefore$ their sum $+ A > M$.

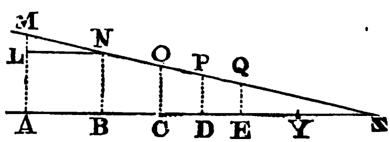
N.B.—The different steps † of this demonstration rest rather on truths that may be implied from Euclid's Elements, than on those expressly taught and proved; they are, however, so plain that, perhaps, a formal proof is not required. The same remark should be extended to the Theorems 3 and 4 which follow.

THEOR. III. *If a series in continued proportion,* A : B : C : D, *&c, be a decreasing one, there is no limit to the diminution of its terms.*

C. 1		Continue the series until a term be found *less than* m, any assigned magnitude however small;
2		let $m : l = B : A$;
3	Use 11, VI.	and let the ratio, $m : l$, be continued in a series;
4	H.	$\because B < A$, $\therefore m < l$, and the series $m : l$ increases;
5		and $\therefore$ a term may be found, $a > A$,
6		Let the series be continued until $a > A$,
7		and let the series A : B : C &c., be continued the same no. of terms;
8		then its last term M will be $< m$, the assigned magnitude.
D. 1	H.	For, $\because A : B : C : D .... : L : M$, $\& \ a : b : c : d \qquad : l : m$;
2	22, V. 16, V.	$\therefore$ *ex. æquo.* $A : M = a : m$; and *alt.* $A : a = M : m$;
3	C. 6. 14, V.	but $A < a$, $\therefore M < m$.
4	*Proced.*	And so on by continuing a like process.

THEOR. IV. "*If a series of magnitudes,* decreasing *in continued proportion, be continued, or imagined to be continued, to an infinite number of terms; the sum of all the terms, or the sum of the series, will be a finite and determinate magnitude.*

C. 1	Use 11, VI.	On the line AZ let the decreasing series be set, AB : BC : CD : DE, &c.,
2	Pst. 1, I.	and draw a line ZM, making an ∠ with AZ;
3	31, I.	at A, B, C, D, E &c. raise parallels cutting MZ in the ·s M, N, O, P, Q;
4	31, I.	draw through · N, NL ∥ AZ, and cutting AM;
5	Conc.	the Sum of the series S = AZ.



Sup. If AZ ≠ S, S is either gr. or less than AZ.

CASE. I. S *is not gr. than* AZ.

D. 1	H.	For each parallel DP, EQ is less than DZ, EZ, from which it is to be taken to determine the · for the next parallel;
2	4, VI.	and $AM : AZ = DP : DZ = EQ : EZ$;
3	14, V.	and $\because AM < AZ$; $\therefore EQ < EZ$.

CASE II. Though the series is unlimited, S *is not less than* AZ.

Sup. If S were $<$ AZ, let S $=$ AY.

D. 1	H.	Now $\because$ the parallels AM, BN, CO, &c., continually decrease proportionally;
2	4, VI.	and $\because$ AZ, BZ, CZ &c. are in proportion to them; $\therefore$ AZ, BZ, CZ &c., are in decreasing continued proportion.
3	Use 11, VI.	Continue this series through a determinate number of terms; at last a term will be found $<$ YZ.
4		Thus the sum of the corresponding parallels must be greater than AY; *i. e.*, AM $+$ BN $+$ CO $+$ &c. are $>$ AY.
5		Hence the Sum of a *limited* no. of terms is $>$ AY, the sum of an *unlimited* number;
6	Ax.	$\therefore$ a part is $>$ the whole, which is absurd;
7	Conc.	$\therefore$ AZ $\ngtr$ S the sum of an infinite number of terms:
8	Case 1.	nor is AZ $\nless$ S, $\therefore$ AZ $=$ S.

3. CERTAIN PROBLEMS also may be deduced from the foregoing principles.

PROB. I. On the last Theorem, Theor. IV. GREGORY'S Problem, as it is named, is to be solved;— *from the two first terms in a series* AB : BC, *to obtain* S, *the sum of the series.*

C. 1	31, I.	Draw NL $\parallel$ AZ, and meeting AM the parallel to BN &c.
2	Sol.	then AZ $=$ S, the sum of the series.
D. 1	2, VI.	For ML : LN $=$ MA : AZ;
2	C. Use 11, VI	but LN $=$ MA $=$ AB; BC $=$ BN; & ML $=$ MA$-$LA, or AB$-$NB, or AB$-$BC;
3	Conc.	$\therefore$ AB$-$BC : AB : AZ;
4	Remk.	*i. e.*, S the sum of the series is a third proportional to the difference of the 1st & 2nd terms, and the 1st term.

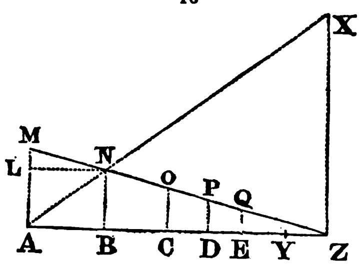
PROB. II. By anticipating Propositions 13 & 16, bk. VI., it follows, that *of the three quantities, the first and second terms,* AB, BC, *&* S *the sum of the series, if any two be given the remaining one may be found.*

1°. Given AB, BC:—then $AZ = \dfrac{AB^2}{AB-BC}$;

as 8 and 4; then $\dfrac{64}{8-4} = 16$

2°. Given AZ & AB; then $AB - BC = \frac{AB^2}{AZ} = BC$.

as 16 and 8; then $8-4 = \frac{64}{16} = 4$.



3°. Given AZ & AB—BC; then $\underline{AB} = \sqrt{AZ \cdot (AB-BC)}$;
as 16 & 8—4; then $\sqrt{16 \times 4} = \sqrt{64} = 8 = AB$;
and $BC = 8-4 = 4$.

4°. Given AZ and BC; then AB : AZ = BC : BZ;
∴ 16, VI. $AB \cdot BZ = AZ \cdot BC$;

Thus, AZ divided at B, so that rect. $AB \cdot BZ$ = rect. $AZ \cdot BC$ will give the solution.

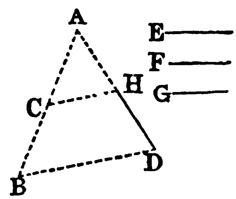
Let AZ, Cor 14, II., be so divided; and the first term of the series will be either segment, AB, or ZB, the Problem having two solutions; for

C. 1	Pst 1, 2, I.	Join AN, and produce it to meet the perp. from Z, in the . X.
2		Then BZ = ZX.
D. 1	2, VI.	∵ AB : BN = AZ : ZX; and AB : AZ = BN : BZ, *i. e.*, BC : BZ;
2	16, V.	and ∵ BZ = ZX; ∴ *alt.* AB : BN = AZ : BZ.
3	Conc.	∴ ZA = S, the sum of the series, of which the first term is BZ, and the second term, BN, or its equal BC.

Prop. 12.—Prob.

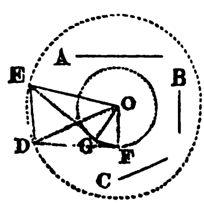
To find a fourth proportional to three given st. lines.

Con. 3, I. 31, I. Dem. 2, VI. 7. V.



E.	1	Data	Let E, F, G, be the three given st. lines;
	2	Quæs.	to find a fourth proportional.
C.	1	Pst. 1, I.	From a com. • A draw AB, AD, forming ∠ BAD;
	2	3, I.	on AB make AC = E, CB = F; and on AD, AH = G; join CH:
	3	31, I.	and draw BD ∥ CH;
	4	Sol.	then HD is the 4th proportional to E, F, G.
D.	1	C. 3, 2, VI.	∵ CH ∥ BD, a side of △ ABD; ∴ AC : CB = AH : HD;
	2	C. 2.	but AC = E, CB = F and AH = G, ∴ E : F = G : HD;
	3	Conc.	∴ to E, F, G, a 4th proportional HD, is found. Q. E. F.

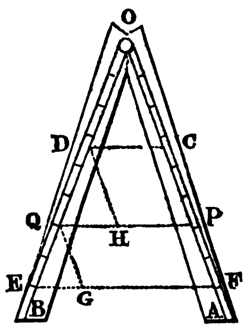
Sch.—Among several other constructions for the Solution of this Problem there is one, for which *the compasses alone* are sufficient,



C.	1	Pst. 3, I.	From any cen. O, with A & B, desc. two ⊙s;
	2	„	& from D, a . in the outer ⊙, set off DE = C;
	3		from D, E, with DO cut the inner ⊙ in G, F;
	4	Sol.	then FG is the 4th proportional required.
D.	1	C. 3, 1.	∵ DF = EG; OE = OD, & OG = OF;
	2	D. 1.	∴ △ODF has its sides = the sides of △OEG;

3	8, I.	∴ △ODF = △ OEG, and ∠ DOF = ∠ EOG;
4	*Add*,or*Sub*.	In ∠s DOF EOG add or take away com. ∠DOG ;
5	Ax. 3, I.	in this case, ∠ EOD = ∠GOF.
6	C.	And ∵ △s GOF, DOE are isosc. & ∠DEO = ∠GFO;
7	Def. 1, VI.	∴ △GOF is eq. ang. to △DOE ;
8	4, VI.	∴ EO : OG = ED : FG ;
9	Conc.	∴ FG is a 4th prop. to A, B, C. Q. E. F.

Use and App. 1—The *Sector* is an instrument invented by Gunter who lived between A.D. 1581 and 1626 ; it consists of two equal legs, or rather rulers, turning like a Carpenter's rule, on a pivot at the centre. The Scales on it OB, OA, converging to the pivot are those which properly belong to the Sector, though in the vacant spaces, it is usual to insert various other scales.



This instrument has not inaptly been termed, "a large number of pairs of compasses packed up into one." For explanation we will take one pair of these compasses or scales, OA, and that which corresponds to it, OB; on each is laid down a scale of chords, OA = OB = 90° ; OP = OQ being radii, or chords of 60° ; and if OP be 4 inches, or 8 inches, and the proper divisions inserted, we have a scale of chords, with a radius of 4, or 8 inches, laid down.

2.—By aid of a pair of Compasses and the Sector, the following Problems, among others, may be solved.

Prob. I.—*To find a fourth proportional to three given lines K* 3, *L* 4, *M* 6

Take OD, OC each = K 3, and open the instrument till DC = L 4 : then take OQ, OP, each = M 6, and the distance QP = the 4th proportional = 8.

For ∵ DC ∥ QP, a side of △OQP ; ∴ OD 3 : DC 4 = OQ 6 : QP 8.

Prob. II.—*To find a chord,* say of 40°, *to a radius,* as of 5 inches, *from a Sector the rad. of the chord of which OP* = 4 *inches.*

From a scale of inches take 5 in the compasses, and open the Sector so that QP = 5 inches; fix now one leg of the compasses at C, the extremity of OC = 40°, and the other leg at D, of OD = 40°; the distance DC will give the chord of 40°. when the radius is 5 instead of 4 inches.

For, by similar $\triangle$s, OP 4 : PQ 5 = OC 40° : DC 50°, when the rad. is 5 inches.

PROB. III.—*To divide a given line L, = 120, into two parts, x and y, which shall be to each other, as two lines, or numbers OD = 3, and OQ = 5.*

Take OE, OF each = (OD 3 + OQ 5) = 8; open the sector so that EF = L = 120; then DC = x, and QP = y.

For, on drawing QG ∥ OF,

$\because$ OD = QE, $\angle$EQG = $\angle$DOC, & $\angle$QEG = $\angle$ODC,
$\therefore$ $\triangle$ODC = $\triangle$QEG, DC = EG, & QP = GF.
Now DC : QP = 3 : 5; $\therefore$ EG : GF = 3 : 5.

Hence the parts will be $x = 3\left(\dfrac{120}{3+5}\right) = 45$; & $y = 5\left(\dfrac{120}{3+5}\right) = 75$.

3.—VARIOUS OTHER LINES are marked ON THE SECTOR, and for their use depend on the same principles as those which affect the Line of Chords and Line of Numbers.

1.—*A Line of Sines*, marked S, of which the rad. = 90°.

2.—*Two Lines of Tangents*, one marked from 0° to 45°, the radius being the sine of 90°;—the other, on a smaller scale, marked from 45° to 75°, the rad. being the tangent of 45°, or distance 0 to the beginning of the scale.

3.—*A Line of Secants*, the rad. being the secant of 0° extending to about 75°

4.—*A Line of Polygons*, marked P O L. showing the length of the sides of regular polygons inscribed in circles.

5.—*A Line of equal parts*, by which a fourth proportional to three given numbers may be found; the line being divided into 100 parts,—that number is the limit of the accurate use of this line.

6.—A *line of chords*, marked C, the radius being the chord of 60°. The extent of this line is from 0 to 60°;—and if a chord for a greater arc than 60° be required,—a circle, with a rad. equal to the chord of 60°, must be drawn, in which the chord of 60° must be set off; and then from the point where the chord of 60° cuts the circle, must also be set off the amount of the given arc above the 60°;—the distance in the circle from the extremities of the sum of the two arcs will be the chord required.

Similar Lines to these, with some additional, are marked on GUNTER'S Scale;—but as this Scale is sometimes of two feet in length, and sometimes of one, it will be found that the Magnitudes, though proportional to the corresponding lines on the six inch Sector are not identical. Besides the Lines of Numbers, Sine Rhumbs. Tangent Rhumbs, &c., on Gunter's Scale, are Logarithmic Lines, the uses of which depends on the principle *that the logarithms of the terms of equal ratios are equidifferent.* For an explanation, of the Logarithmic Lines on Gunter's Scale, their Construction and Use, the Student may consult KEITH'S *Plane and Spherical Trigonometry*, pp. 18—24. The *General Rule* given is this; "the extent of the compasses from the first term to the second, will reach, *in the same direction*, from the third to the fourth term. *Or*, the extent of the compasses from the first term to the third, will reach, *in the same direction*, from the second to the fourth."

4.—An Example or two will show USE OF THE SECTOR.

Ex. 1. *In a rt.* $\angle$*d.* $\triangle ADC$, *as in fig. to* 13, *VI, to find the side DC opposite* $\angle A$, *when* $\angle A = 40°$ *and* $AC = 4$ *inches.*

Take 4 inches in the compasses, and open the Sector, until 90° and 90° on the two identical lines of Sines are 4 inches apart; then the distance from 40° on one of those lines of Sines to 40° on the other will equal the side DC.

Ex. 2. *Given in a* $\triangle$ *two* $\angle s$, *of* 59° *and* 38° *respectively, and the side opposite to* $\angle 59°$ *equal to* 76 *equal parts;—required the length of the side opposite to* $\angle 38°$.

The analogy is,—$\text{Sine } 59° : \text{Sine } 38° :: 76 : x$; *i. e.*, $x = \dfrac{\text{Sine } 38° \times 76}{\text{Sine } 59°}$.

Open the Sector until the counterpart lines of sines, 59° and 59°, are as far apart as the distance from 0 to Sine 38°; then the distance from 76 to 76, on the corresponding lines of equal parts, will show the value of x.

Ex. 3. *To inscribe a regular polygon of* 10 *sides in a circle of which the radius is* 3 *inches.*

Open the Sector until 6 and 6, on the counterpart lines of Polygons, are 3 inches apart, then the distance from 10 to 10, on the same counterpart lines, will equal the length of one side.

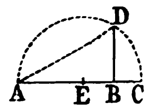
5.— The Sector may also be applied, to extract the Square and Cube Roots, and to double the Cube; to measure Triangles,—to find the Areas of Figures, and the Contents of Solids,—and to increase or diminish any Figure according to any given Proportion.

NB. It must however be remembered, that no great accuracy is to be obtained from the small Sectors in common use; and they are the less to be trusted in proportion to the greater opening of the sides or legs. It is only by considerable practice that a person can become expert and exact in the use of this instrument.

PROP. 13.—PROB.

To find a mean proportional between two given st. lines.

CON. Pst. 3, I, 11, I. DEM. 31, III. COR. 8, VI.



E. 1	Data	Let the given st. lines be AB and AC;
2	Quæs.	to find a mean proportional between them.
C. 1	Pst. 3, I.	Place AB, BC, in a st. line AC; & on AC descr. a semicircle ADC;
2	11, I.	from B draw BD ⊥ AB and join AD, DC;
3	Sol.	then BD is a mean proportional to AB & AC.
D. 1	31, III. C. 2.	∵ ∠ADC is a rt. ∠, & in the rt. ∠'d △ABC, BD is ⊥ the base;
2	Cor. 8, VI.	∴ DB a mean propl. to the segments of AC, namely, AB, BC;
3	Conc.	∴ between AB, BC, a mean proportional DB, is found. Q. E. F.

SCH. There are other constructions by the aid of which a mean proportional between two given lines can be obtained,—but none simpler or easier than the above. For *numerical* calculations however the following formula may be used: Let $AC = 2r$; $AB = x$; $BC = 2r - x$; and $BD = y$

then $x : y : 2r - x$; or, $y^2 = 2rx - x^2$, & $y = \sqrt{2rx - x^2}$,

or $y = \sqrt{AB \cdot BC}$.

EX. Find a mean proportional to 9 and 4.

$\sqrt{4 \times 9} = \sqrt{36} = 6$ the mean propl; *i. e.*, $9 : 6 = 6 : 4$.

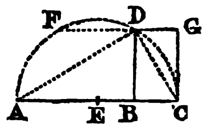
USE and APPL. I.—*Any rectangular parallelogram may be reduced to an equivalent square by this proposition*;—for if AB & BC represent the two sides, then the square on DB = AB · BC;—the demonstration of which follows from Prop 17, VI.

II. Combining Propositions 11 and 13, we are able, *when, of three lines in continued proportion, any two are given, to find the unknown line*; See Use and Appl. 11, VI. Problem II.

III. *Given one of three terms and the sum of the other two; to find the two unknown terms.*

1°. *Given the mean* BD, *and the sum* AC = AB + BC.

C. 1	Pst. 1, I. 11, I.	On AC desc. a semi. ⊙; and at C draw CG, = BD ⊥ AC;
2	31, I.	Through G draw GF ∥ AC, cutting the arc in · s F, D;
3	12, I.	and from D drop DB ⊥ AC;
4	Sol.	then the other two terms are AB and BC.

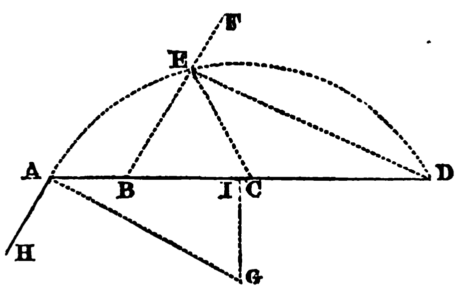


D. 1	C. 4, VI.	∵ △s ABD and DBC are similar, ∵ AB : BD = BD : BC;
2	Conc.	Thus BD being the mean, AB & BC are the extremes.

Q. E. F.

2°. *Given* AB, *one extreme, and the sum of the two terms,* BD = BC + CD; *required the mean, and the other extreme.*

C. 1	2, I.	Set the sum of the three terms, AB + BD, in one st. line AD;
2	23, I.	at the extremity A draw an ∠, DAH = ext. ∠ of an eq. lat. △;
3	33, III.	and on AD desc. a seg. of a ⊙ with ∠ = ∠ DAH;
4	3, I.	then set on AD the given extreme AB;
5	23, I,	at B make ∠ ABE = ∠ DAH;
6	23, I,	prod. BE, and at E make ∠ FEC = ∠ DAH;
7	Sol.	BC is the mean, and CD the other extreme.

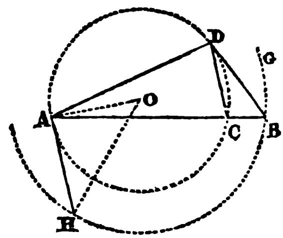


D. 1	C.	∵ EBC is an eq. lat. △, and BC = BE;
2	Sch. 1. 8, VI.	∴ AB : BC = BC : CD.
3	Conc.	∴ BC is the mean, and CD the other extreme.

Q. E. F.

IV. **When *three lines* are in continued proportion, *if one be given and the difference of the other two, those other two may be found.***

1°. **Given the mean BD, and AC the difference of the extremes.**

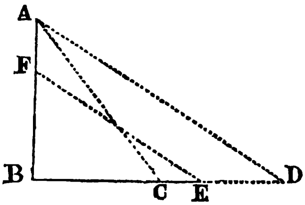


C. 1	33, III.	On AC descr. any seg. of a ⊙ ADC;
2	17, III.	To the ⊙ at A draw a tang. AH = given mean;
3	Psts 1 & 3, I.	join H and cen. of the ⊙, and with OH desc. concentric ⊙ HBG;
4	Psts. 2 & 3, I.	produce AC to B; and from B with rad. = AH cut the inner circle in D;
5	23, I.	join DA and DC;
6	Sol.	then AB and BC are the two terms required;
D. 1	C.	∵ BD touches the ⊙ ACD in D, and DC cuts it in C;
2	32, III.	∴ $\angle$ BDC = $\angle$ DAC.
3	C. & D. 2	In $\triangle$s ADB, DCB we have $\angle$BDC = $\angle$DAC, and $\angle$B com.
4	Def. 1 VI	∴ $\triangle$s ADB, DCB, are eq. ang., and their sides proportional;
5	4, VI.	∴ AB : DB = DB : BC.
6	Conc	∴ AB and BC are the other two terms.

2°. *Given one extreme,* BC, *and the difference between the mean and the other extreme,* AB—BD; *to find both the mean and that other extreme.*

D. 1	Theo. I. Use 11, III.	∵ the differences of successive terms are as the terms themselves,
2	16, VI.	∴ the rect. under one extr. and the dif. of the mean and the other extr. = the rect. under the mean and the difference between it and the given extreme.
3	Remk.	*i. e.*, BC. (AB—DB) = DB. (DB—BC).
4	D3. & H.	Since the Area of this rect. and the difference of the sides are given;
5	Use 3, 10, II.	∴ the sides themselves may be found.
6	Remk.	For $\sqrt{\left(\frac{\text{difference}}{2}\right)^2 + \text{the area}} = \frac{\text{Sum}}{2}$. LARDNER's Euclid, p. 79.
7	Use 1, 10, II.	and $\frac{\text{Sum}}{2} \pm \frac{\text{Diff.}}{2}$ = the greater, or the less magnitudes.

V. *To find two st. lines to contain a rectangle equal to a given rectangle* **AB . BC,** *and to have a given ratio one to the other.*



C. 1	*Pon.*	Place AB and BC so as to be conterm. at $\angle$ B;
2	12, VI.	prod. BC to D so that AB is to BD in the given ratio;
3	13, VI.	join AD, and take BE a mean propl., *i. e.*, BC : BE = BE : BD,
4	31, I.	and through E draw EF $\parallel$ DA, meeting AB in F;
5	Sol.	the lines BE, BF, are the st. lines required.
D. 1	C. 2, 3.	Now the lines are evidently in the given ratio;
2	2, VI.	and $\because$ EF $\parallel$ DA, $\therefore$ AB : BD = FB : BE;
3		and $\therefore$ also BF : BA = BE : BD;
4	20, VI. C. 3.	$\therefore$ BF . BE : BA . BD = BE^2 : BD^2, *i. e.* BC : BD;
5	1, VI.	but AB . BC : AB . BD = BC : BD,
6	9, V.	$\therefore$ EB . BF = AB . BC.
7	Sol.	$\therefore$ BE and BF are the two required lines. Q. E. F.

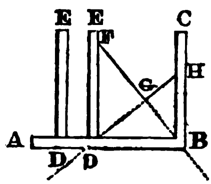
VI. We are also able by the 13th Prop. to find any number of means represented by a power of 2, *minus* 1; *i. e.* 4—1 or 3, 8—1 or 7, 16—1 or 15 &c. For having obtained *one* mean between two magnitudes, we may by the same process determine a mean between it and each of its extremes, and thus we shall have *three* means. And again between each successive pair of the series thus found inserting means, and pursuing the same method, we obtain *seven* means, *fifteen*, *thirty one*, &c.

Addenda to Prop. 13, VI.

I.—The Problem, *to obtain* two mean proportionals *between two given st. lines*, on which depends "the duplication of the cube" in Solid Geometry, cannot be determined by the rule and compasses, the only instruments allowed by Euclid; and consequently it has never received a strictly Geometrical Solution. Various mechanical contrivances have been invented to attain the solution; as Plato's Method of a ruler inserted perpendicularly on the side of a square and moveable along that side; Philo's of Byzantium, of a graduated ruler revolving on a point; the Trammel of Apollonius, or of Nicomedes who lived

about two centuries before the birth of Christ; and the Method of **Des Cartes**, consisting of a collection of rulers, two of which move round a pivot, each of the two having a groove on the inner edge, in which other rulers are free to move perpendicularly to the groove. For a description of these Methods reference may be made to **Lardner's** *Euclid* pp 196-199 **Cooley's** *Supplement to Euclid* pp 92, 93; or *Galbraith's Manual* pp. 111, 112. There are several solutions in the *third* Bk. of the Mathematical Collections of **Pappus**, who flourished A.D. 379—395; and ten different solutions occur in a Commentary by **Eutocius** of Ascalon, who lived A.D., 560, *on the Sphere and Cylinder of* **Archimedes**.

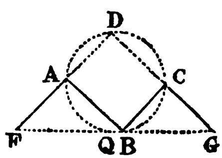
1°. **Plato's** Method *of a perpendicular moveable along the side of a Square.*



C. 1	Take two st. rulers AB, BC, fitted at a rt. ∠ ABC;
2	and a third ruler moving along AB at rt. ∠s to AB;
3	place the given extremes FG and GH at a rt ∠ FGH;
4	and so that FG produced cuts the rt. ∠ABC in the vertex B;
5	while HG produced cuts BA in D,
6	and the perp. DE meets the . s D & F;
7	then GB and GD are the two mean proportionals to FG & GH.

D. 1	C.	∵ HBD & FDB are rt. ∠d △s of which the altitudes are GB, GD;
2	8, VI.	∴ HG : GB : GD, and BG : GD : GF.
3	Conc.	∵ GB & GD are mean proportionals to the extremes HG & GF.

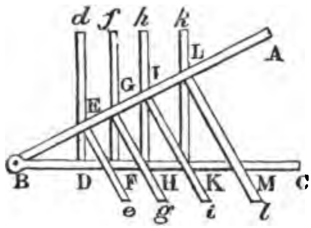
2°. **Philo's** Method *of a graduated ruler FG revolving round, B, the vertex of a rt. angle.*



C. 1	11, I.	Place the given extremes, AB, BC, at a rt ∠ at B;
2	31, I. 4, IV.	complete the rect. ABCD, and about it descr. a ⊙ ABCD;
3	Pst. 2, I.	produce DA & DC; and move FG about the . B, until QF = BG;
4	Conc.	then AF & CG are the two means to AB & BC.
D. 1	C.	∵ BG = QF, & QG = BF; ∴ rect. QG . GB = rect. QF . BF;
2	36, III.	but QG . GB = DG . GC; & QF . BF = DF . FA;
3	Ax. 1.	∴ DG . GC = DF . FA;

4	16, VI.	$\therefore$ DG : DF = AF : GC;
5	C.	but $\because$ $\triangle$s GDF, BAF & GCB are equiangular,
6	2, VI.	$\therefore$ DG : DF = AB : AF, & DG : DF = GC : CB;
7		and $\therefore$ BA : AF = AF : GC;
		and AF : GC = GC : CB.
8	Conc.	$\therefore$ AF & CG are the two means to AB & CB.

3°. Method of Des Cartes, *with a collection of rulers.*



C. 1		Take two st. rulers AB, BC united by a pivot at B;
2		in each there is a groove, in which rulers move perpendicularly to AB & BC;
3		so that perp. D*d*, on opening the rulers, pushes forward E *e*, E *e* pushes F *f* &c.
4	*Sup.*	*Let two means be required.*
5		Move D*d* from B, until BD = the less extreme;
6		close AB upon CB that the perpendicular may move up to D,
7		and fix D *d* in the position where BD = the less extreme;
8		Now open AB & CB, until BG = the greater extreme, D*d*, during the process, pushing E *e* from B, & E *e* pushing F *f*, &c.;
9	Sol.	then the two means are BF & BE.
D. 1	C.	$\because$ $\triangle$s BDE & BFG are equiangular;
2	4, VI.	$\therefore$ BD : BF = BF : FE; & BF : FE = FE : BG;
3	Conc.	$\because$ BF & BE are the two means to BD & BG

In a similar way *three* means BE, BF, BG will be found to BD & BI, by opening the rulers AB, CB, until BH = the greater extreme.

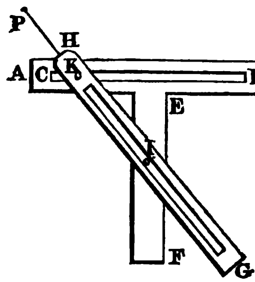
For *four* mean proportionals open the rulers until BI = the greater extreme; and so on for a greater number of means.

Generally, if an *even* number of means be required, the extremes will be on different rulers; if an *odd* number, on the same ruler.

N.B.—The advantage of the Method of Des Cartes is more apparent than real. "It would appear that any number of mean proportionals may be found; but the necessary adjustments are almost as deficient in practical facility as in geometrical legitimacy."—Cooley.

II. The *Trisection of a rectilineal angle*, ABC, is another puzzle for Plane Geometry. The Solution of this Problem, though of no more trouble by aid of the Higher Analysis than the finding of the cube root, is impossible by the circle and right line. NICOMEDES, known as the discoverer of the conchoid curve, effected the Solution by the *Trammel* which he invented, and by which the Duplication of the Cube and the Trisection of an angle are both accomplished.

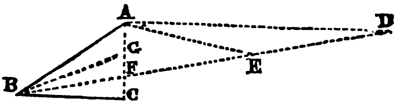
1°. The TRAMMEL of NICOMEDES, *a* T *square with a moveable ruler* HG.



C. 1 Take a T square with a groove CD, in the cross ruler AB; and a fixed pin, I, in the stem, FE;
2 Let this fixed pin be inserted in the groove of the moveable ruler HG;
3 and HG also have a fixed pin K, inserted in CD.
4 From H, in HG, there issues a sliding stem HP;
5 Its length HP = that part of the line to be intercepted by the sides of the angle.
6 Let the fixed pin I be now placed on the given point;
7 and the groove CD on one side of the given angle;
8 let HG be moved so that the pin K go along one side of the angle;
9 and let the motion be continued until the · P shall come on the other side of the angle;
10 then the line joining P & I, *i. e.*, PI, is the required line.

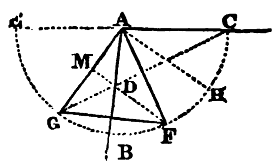
2°.—By this instrument *to Trisect a given angle ABC.*

C. 1	12, I. 31, I.	From A draw AC $\perp$ BC, and AD $\parallel$ BC;
2	Trammel.	inflect BD so that FD = 2 BA;
3	10, I.	bisect FD in E, and join AE;
4	Sol. 1.	then $\angle$DBC = $\frac{1}{3}$ $\angle$ABC;
5	Sol. 2.	bisect $\angle$ABD by BG, & $\angle$ABC is trisected.



D. 1	C. I. 39, I.	$\because$ $\angle$FAD is a rt. $\angle$; and AE $=$ ED $=$ EF ;
2	5, I. 32, I.	$\therefore$ $\angle$EAD $=$ $\angle$ADE ; & ext. $\angle$AEB $=$ 2 $\angle$ADB ;
3	29, I.	but $\angle$ADB $=$ $\angle$DBC ; $\therefore$ $\angle$AEB $=$ 2 $\angle$DBC ;
4	C. 2. 5, 1.	and AE $=$ AB ; $\therefore$ $\angle$AEB $=$ $\angle$ABE,
5	D. 2 & 3.	$\therefore$ $\angle$ ABE $=$ 2 $\angle$DBC ; & $\therefore$ $\angle$FBC $= \frac{1}{3}$ $\angle$ABC.
6	D. 5. C. 5.	And $\because$ $\angle$ ABD $= \frac{2}{3}$ $\angle$ABC, and is bisected ;
7		$\therefore$ $\angle$CBF $=$ $\angle$FBG $=$ $\angle$GBA ;
0	Conc.	$\therefore$ $\angle$ABC is trisected by the lines BF, BG. Q, E. F.

3°.—*Or*, by Cooley's Tentative Method, *to trisect a given* $\angle$*BAC.*



C. 1	Pst. 2, I.	Produce the side, as A C beyond A, the vertex ;
2	Pst. 3, I	from A descr. a semi $\odot$ EBC ;
3	*Superp.*	on the . A place the vertex of an eq. lat. $\triangle$, made of card or board ;
4		and let the $\triangle$ move round the . A on a pin.
5		At M, the middle · of AG, a side of the $\triangle$, or of the part between A & the circumference, fix a fine thread ;
6		and attach the thread to F ;
7	Sim.	So, fix a thread at G; and keep it extended through C.
8		Now, if the eq. lat. $\triangle$ move round A till the intersection in D of the two threads falls on the line AB,
9		then AF will cut off $\frac{1}{3}$ of $\angle$CAB.
D. 1	C. Sch. 32I.	$\because$ fig. AGF, is an eq. lat. $\triangle$ and $\angle$GAF $= \frac{1}{3}$ of 2 rt. $\angle$s;
2	13, I.	$\therefore$ $\angle$GAF $= \frac{1}{3}$ of the $\angle$s EAG, GAF, FAC ;
3	D. 2.	$\therefore$ $\angle$GAF $= \frac{1}{2}$ of $\angle$s EAG $+$ FAC.
4	20, III.	But $\angle$EAG at centre $=$ 2 $\angle$ACG at $\odot$ce.
5		& $\because$ $\angle$ACG $=$ $\angle$FAB, $\therefore$ $\angle$EAG $=$ 2 $\angle$FAB, *i. e.*, $\angle$FAB $= \frac{1}{2}$ $\angle$EAG ;
6	26, III.	Now $\because$ $\angle$FAB $= \frac{1}{2}$ $\angle$FAC : $\therefore$ arc BF $= \frac{1}{2}$ arc, FC ;
7		$\therefore$ arc BF $= \frac{1}{3}$ arc BC, & $\angle$BAF $= \frac{1}{3}$ $\angle$BAC ;
8	Conc.	$\therefore$ $\angle$BAC is trisected by AF and AH. Q. E. F.

N.B.—Either of these methods solves the Problem, and may be adopted in practice when the Trisection of an angle is required.

Prop. 14.—Theor.

Equal Parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, *parallelograms that have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional, are equal to one another.*

"When ▱s are eq. ang. the sides which are about the eq. ∠s, are reciprocally proportional; and when the ▱s are eq. ang., and the sides, which are about the eq. ∠s, are reciprocally proportional, those ▱s are equal."—Euclid.

Con. 14, I If at a · in a st. line, two other st. lines, upon the opp. sides of it, make the adj. ∠s together eq. to two rt. ∠s, these two st. lines shall be in one and the same st. line.

31, I. To draw a st. line through a given · ∥ to a given st. line.

Dem. Def. 2, VI. Reciprocal △s and ▱s have their sides about two of their ∠s proportionals, so that a side of the 1st fig : a side of the other : : the rem. side of the other : the rem. side of the 1st.

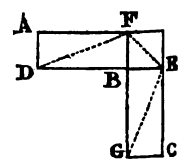
7, V. Eq. Ms have the same ratio to the same M; and the same has the same ratio to eq. Ms.

1, VI. △s and ▱s of the same alt. are to one another as their bases.

11, V. Ratios that are the same to the same ratio, are the same to one another.

9, V. Ms which have the same ratio to the same M are eq. to one another: and those to which the same M has the same ratio are eq. to one another.

E. 1	H. Case I.	Let ▱ AB = ▱ BC, & ∠DBF = ∠ EBG
2	Conc.	then DB, BF and GB,BE are reciprocally
	Def. 2. VI.	proportionals. *i.e.*, DB : BE = GB : BF.
C. 1	*App.*	Apply ▱ AB to ▱ BC so that DB, EB form one st. line;
	14, I.	and FB, BG also one st. line:
2	31, I.	and complete the ▱ FE.



D. 1	H. & C.	∵ ▱ AB = ▱ BC, & EF is another ▱;
2	7. V.	∴ ▱ AB : ▱ FE = ▱ BC : ▱ FE;
3	1, V.	but ▱s AB : FE = bases DB : BE, and ▱s BC : FE = bases GB : BF;
4	11, V.	∴ sides DB : BE = sides GB : BF;
5	Conc.	∴ the sides of ▱s AB, BC, about the eq. ∠s are reciprocally proportionals.
E. 1	H. Case II.	Let sides DB : BE = GB : BF;
2	Conc.	then ▱ AB = ▱ BC.
D. 1	H. & 1, VI.	∵ DB : BE = GB : BF; and DB : BE = ▱ AB : ▱ EF;
2	1, VI.	and ∵ GB : BF = ▱ BC : ▱ EF;
3	11, V.	∴ ▱ AB : ▱ FE = ▱ BC : ▱ FE;
4	9, V.	∴ ▱ AB = ▱ BC.
5	Rec.	Therefore, *Equal parallelograms, &c.* Q.E.D.

Sch. 1. There is a *third* case, not introduced in the proposition, and depending for its proof on Prop. 16, bk. VI;—it is, "*When two ▱s have eq. areas and their sides are reciprocally propl., they will be eq. ang.*" It will be sufficient to prove that one ∠ in the one is eq. to the corresponding ∠ in the other ▱; for from the proof it follows, that each separate ∠ in the one is eq. to each corresponding ∠ in the other.

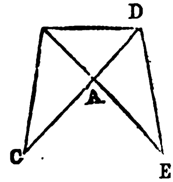
2. Inasmuch as △s = the halves of their equiangular ▱s,—what is proved respecting eq. ▱s might be proved respecting eq; △s. (Ax. 5, I); and the following Proposition, the 15th, form a corollary of the 14th; for if △DBF = △ EBG, and ∠ DBF, = ∠EBG, then also DB : BE = GB : BF. The parallelogram and triangle are united into one Theorem, Prop. 1, VI, and might also be united here.

Prop. 15.—Theor.

Equal angles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally pro-

portional: and conversely, *triangles which have one angle in the one equal to one angle in the other, and the sides about the equal angles reciprocally proportional, are equal to one another.*

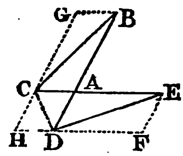
Con. 14, Pst. 1, I. Dem. 7, V. 1, VI. 11, V. Def. 2, VI. 9, V.



E. 1	Hyp.	Case I.—Let $\triangle ABC = \triangle ADE$, and $\angle BAC = \angle DAE$;
2	Concl.	then CA, AB, EA, AD are reciprocally proportional.
	Def. 2, VI.	*i. e.*, CA : AD = EA : AB.
C. 1	*Appl.*	Place CA & AD in one st. line CD;
2	14, 1. Pst. 1, I.	∴ EA, AB also in one st. line EB; join DB.
D. 1	H. & C.	∵ $\triangle ABC = \triangle ADE$, & fig. ABD is another $\triangle$;
2	7, V.	∴ $\triangle ABC : \triangle ABD = \triangle AED : \triangle ABD$.
3	1, VI.	but $\triangle ABC : \triangle ABD$ = bases CA : AD, and $\triangle AED : \triangle ABD$ = bases EA : AB;
4	11, V.	∴ CA : AD = EA : AB.
5	Conc.	∴ the sides about the eq. $\angle$s are reciprocally proportional.

E. 1	Hyp.	Case II.—Let CA : AD = EA : AB;
2	Conc.	then $\triangle ABC = \triangle ADE$.
C.	14, I.	Make the same construction as in Case I.
D. 1	H.	∵ CA : AD = EA : AB;
2	1, VI.	and ∵ CA : AD = $\triangle ABC : \triangle ABD$, and EA : AB = $\triangle ADE : \triangle ABD$;
3	11, I.	∴ $\triangle ABC : \triangle ABD = \triangle ADE : \triangle ABD$;
4	D. 3. 9, V.	and ∵ $\triangle ABD$ is com., ∴ $\triangle ABC = \triangle ADE$.
5	Rec.	Therefore, *equal triangles which have one angle, &c.* Q. E. D.

Or, for Case II.



C. 1	*Appl.*	Place as before, CA, AE in one st. line, and DA, AB in another.
2	31,	and complete the ▱s AF, AG, AH.
D. 1	H.	∵ BA : AD = EA : AC,
2	1, VI.	∴ BA : AD = ▱ AG : ▱ AH; and EA : AC = ▱ AF : ▱ AH;
3	9, V. 41, I.	∴ ▱ AG = ▱ AF, and △ ABC = △ ADE.

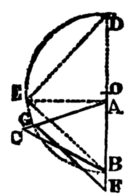
N. B, Case I, may be proved on the same construction.

Schol. 1. As a *criterion* for the equality of triangles this Proposition may be classed with Props. 4, 8, & 26, bk. I.

2. When in each triangle the equal angle, included by the reciprocal sides has the same supplement, the sides are also reciprocally proportional.

Use & App. *To construct an isosceles triangle equal to a given scalene triangle* A B C, *and with the same vertical angle* B A C.

C. 1	3, I.	Produce BA, and make AD = AC;
2	Pst. 3, I. 11, I.	on BD desc. a ⊙ DEB; and draw AE ⊥ BD;
3	3, I.	in AB prod., take AF = AE, and in AC, AG = AE;
4	Sol.	join GF; then △ AGF is the isosc. △ required.
D. 1	Cor. 8, VI.	∵ BA : AE = AE : AD, ∴ BA : AE = AE : AC;
2	C. 3, Def. 25, I.	but AF, AG, each = AE; ∴ AGF is an isosc. △
3	D. 1, 2.	and ∴ BA : AG = AF : AC. *i. e.*, the sides about the vert. ∠ are recip. props.
4	15, VI.	∴ the isosc. △ AGF = the scalene △ ABC.



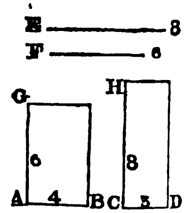
Q. E. F.

Prop. 16.—Theor. (*Very Important.*)

If four st. lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and conversely, *if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four st. lines are proportionals.*

Con. 11, I. 3, I. 31, I.

Dem. 7, V. 14, VI. Def. 1, II. A rect. is contained by any two of its conterminous sides. Ax. 1, I.



E. 1	Hyp.	Case I.— Let AB : CD = E : F;
2	Conc.	then ▭ AB . F = ▭ CD . E.
C. 1	11, I.	From A, C, draw AG ⊥ AB, and CH ⊥ CD,
2	3, I. 31, I.	make AG = F, CH = E, and complete ▭s BG, DH.
D. 1	C. 1 & 2.	∵ AB : CD = E : F, and ∵ E = CH and F = AG.
2	7, V.	∴ AB : CD = CH : AG; *i. e.*, sides about the eq. ∠s are recip. propl.;
3	14, VI.	∴ ▭ BG = ▭ DH;
4	C. 2, Def. 1, II.	but ∵ AG = F, ∴ ▭ BG is contained by AB and F;
5		and ∵ CH = E, ∴ ▭ DH is contained by CD and E;
6	Conc.	∴ ▭ AB . F = ▭ CD . E.

E. 1	Hyp.	Case II.—Let ▭ AB . F = ▭ CD . E
2	Conc.	then AB : CD = E : F.
C.	*Sim.*	Using the same construction;
D. 1	H. & C.	∵ AB . F = CD . E, and ∵ AG = F and CH = E;
2	Def. 1, II.	∴ ▭ BG is contained by AB . F, and ▭ DH by CD . E;

3	Ax. 1.	$\therefore \square BG = \square DH$.
4	C. 14, VI.	and $\because \angle A = \angle C$, $\therefore AB : CD = CH : AG$, *i. e.*, they are reciprocally proportional;
5	C. 7, V.	but $CH = E$, and $AG = F$; $\therefore AB : CD = E : F$.
6	Rec.	$\therefore$ *If four st. lines be proportionals &c.* Q. E. D.

Alg. & Arith. Hyp. Let $a\ 4 : b\ 3 = c\ 8 : d\ 6$.

Alg. $\because \frac{a}{b} = \frac{c}{d}$,

$(\times b \times d)$ $\frac{abd}{b} = \frac{bcd}{d}$, i. e., $ad = bc$;

conversely, $\because ad = bc$.

$(\div b \times d)$ $\frac{ad}{bd} = \frac{bc}{bd}$, i. e., $\frac{a}{b} = \frac{c}{d}$.

Arith. $\because \frac{4}{3} = \frac{8}{6}$,

$(\times 3 \times 6)$ $\frac{4 \times 3 \times 6}{3} = \frac{8 \times 3 \times 6}{6}$, *i. e.*, $4 \times 6 = 3 \times 8$;

conversely $\because 4 \times 6 = 3 \times 8$.

$(\div 3 \times 6)$ $\frac{4 \times 6}{3 \times 6} = \frac{3 \times 8}{3 \times 6}$, *i. e.*, $\frac{4}{3} = \frac{8}{6}$.

COR. *Rectangles which have their sides about the right angles reciprocally proportional are equal; and if the rectangles are equal, the sides about the right angles are reciprocally proportional.*

SCH. I. DEFINITION. Two ratios A : B and $a : b$ are said to be reciprocally proportionals, when the antecedent is to the consequent in the one as the consequent to the antecedent in the other; as $A : B = b : a$.

II. As a necessary consequence, the 16th Prop., Bk. VI. may be deduced from the principles resting on this definition;—1°. That a ratio compounded of reciprocal ratios is a ratio of equality; and 2°. That if a ratio of equality be compounded of two ratios they must be reciprocals.

1°. *A ratio compounded of reciprocal ratios, as,* A : B *and* $a : b$, *is a ratio of equality.*

D. 1	H.	$\because A : B = b : a$; $\therefore \left.\begin{matrix} A : B \\ a : b \end{matrix}\right\} = \dfrac{A : B}{B : A}$.
2	Conc.	$\therefore A : b = A : A$; and $A : A$ is a ratio of equality.

N. B. The form $\left.\begin{matrix} A : B \\ a : b \end{matrix}\right\}$ denotes a ratio compounded of $A : B$ and $a : b$.

2°. *If a ratio of equality, $A : A$, be compounded of two ratios, $a : b$, and $c : d$, they must be reciprocals.*

D. 1	C.	Let $c : d = b : x$;
2		then $\left.\begin{matrix} a : b \\ c : d \end{matrix}\right\} = \left\{\begin{matrix} a : b \\ b : x \end{matrix}\right\}$, or $a : d = a : x$;
3	H.	but $a : x$ is a ratio of equality; *i. e.* $x = a$.
4	Conc.	$\therefore b : x$ is the the recip- of $a : b$; and $c : d$ is the recip. of $a : b$.

3°. *The sides of equal rectangles are four proportionals.*

D. 1	Cor. 2, 1, VI.	∵ △s and ▱s are in a ratio compounded of the bases and altitudes.
2	Theor. 2 Sch. II.	∴ if the △s and ▱s are equal, those bases and alts. are recip. propls.,
3	Theor. 2 Sch. II.	and also those △s or ▱s are equal,
4	Remk.	Now, ∵ the means are sides of one, and the extremes the sides of the other;
5		∴ those sides of the eq. ▱s are propls.;
6		and ∵ the sides are propls.
7	Conc.	∴ the ▭ under the means = ▭ under the extremes;
8	16, VI.	and ∴ the sides of those eq. ▭s are proportionals.

"Thus the sixteenth Proposition," says LARDNER, p. 203, "follows immediately from the first."

III. By the principle, that if four st. lines are proportional, the rectangle of the extremes is equal to the rectangle of the means, *we convert the equality of the two ratios*, or the proportion of the four lines, *into the equality of two rectangles.* In numerical proportion the expression, *the product of the extremes equals the product of the means*, points out a very similar, if not an identical process; and the phrase, "product of two lines," is sometimes used instead of "rectangle under two lines." The word *product* however, supposes the four magnitudes which are proportionals to be commensurable, and really denotes the result by multiplication of the numbers which stand for the magnitudes: but in general, it may be demonstrated both of *commensurable* and of *incommensurable* magnitudes, that, *if four st. lines*, A, B, C, D *be proportionals*, A : B :: C : D, *the rectangles under the extremes* A . D = *the rectangle under the means*, B . C.

IV. *The doctrine of* LIMITS, however, is needful for the *full* demonstration of this Proposition, when incommensurable magnitudes are introduced. Briefly stated, by a *limit* we mean a *fixed* magnitude to which another and a *variable* magnitude may approach as nearly as we please; and yet the variable magnitude is never able to attain an exact equality with the fixed magnitude. Thus, the circle is the limit of an inscribed polygon; the polygon may approach in area, as near as we choose, to the area of the circle, but never actually attains that area. Take any fixed quantity, A, and a variable quantity P;—in order that A may be named the limit of P, the two conditions have to be fulfilled—1st, that P never attains equality to A;—and 2nd, that P shall be capable of being made as nearly as we please on an equality with A. Again,

"Two fixed magnitudes, A and B, are the limits of two others, P and Q, when P and Q by increasing together, or by diminishing together, may be made to approach more nearly to A and B respectively, than by any the same given difference, but can never become equal to, much less pass A and B."—GEOM. *Plane, Sol. & Spher.* p. 248.

V. For the proper elucidation of this subject two leading Theorems are needed, given as M, V. p. 235; and P. V. p. 241.

THEOR. I. *If there be two fixed magnitudes* A and B *which are the limits of two others,* P and Q, *and if P be to Q always in the same given ratio of C to D; then A shall be to B in the same ratio.*

E. 1	Hyp. 1.	CASE 1. *Let P and Q, approach A and* B *by a continual increase.*
2	„ 2.	Let P and Q never $=$ A and B, much less exceed A and B,
3	Hyp. 3.	but P & Q approach A and B more nearly than by any given difference.
4		Let now a M, B′ be taken, so that A : B′ $=$ C : D;
5		if B′ $\neq$ B, B′ $>$ or $<$ B.

SUP. 1°. *Let B′ be $<$ B, by any difference, as b;*

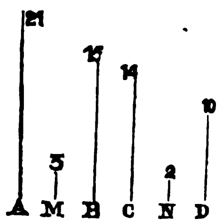
D. 1	H. 11, V.	$\therefore$ P : Q $=$ C : D, & A : B′ $=$ C : D; $\therefore$ A : B′ $=$ P : Q;
2	H.	but A always $>$ P, $\therefore$ B′ always $>$ Q.
3	D. 2, & Sup. 1°.	$\therefore$, $\because$ Q always $<$ B′, and B′ $<$ B by b;
4		$\therefore$ Q cannot approach B within the difference b;
5		but this is against the Hyp,; $\therefore$ B′ $\not<$ B.

SUP. 2°. *Let B′ $>$ B; and take A′ so that A′ : B $=$ A : B′.*

D. 1	H. 14, V.	$\because$ B < B′ $\therefore$ A′ < A, as by some difference *a*.
2	Sup. 2°. H. 11, V.	And $\because$ A′ : B = A : B′ & P : Q = A : B′; $\therefore$ A′ : B = P : Q;
3	H.	but B always > Q ; $\therefore$ A′ always > P ;
4		$\therefore$, $\because$ P always < A′, & A′ < A by *a* ;
5		$\therefore$ P cannot approach A within the difference *a*;
6		but this is contrary to the Hyp. ; $\therefore$ B′ $\not>$ B ;
7	Conc.	$\therefore$ B′ cannot but = B, *i. e.* A : B = C : D
E.	H.	CASE II. Let P & Q approach A & B respectively by a continual decrease.
D. 1	*Sim.*	In this case the same demonstration may be given, taking care to substitute the word "greater" for "less" and "less" for "greater."
2	Rec.	Therefore; *if there be two fixed magnitdues, &c.* Q. E. D.

THEOR. II. *If four st. lines,* A B C D, *be proportionals.* (whether commensurable or otherwise) *the rectangle under the extremes will be equal to the rectangle under the means.*

CASE I. Let A and B be *commensurable,* & $\therefore$ also C & D.



E. 1	Hyp.	Let their com. R. be 7 : 5, and their com. meas. M and N;
2	Pst. 2, V.	*i. e.* let M be contained 7 times in A, and 5 times in B,
3		and N be contained 7 times in C, and 5 times in D;
D. 1	E. 2, 3.	$\because$ A = 7 M, and D = 5 N; $\therefore$ A . D = 7 × 5 times M.N;
2	E. 2, 3.	and $\because$ B = 5 M, and C = 7 N; $\therefore$ B . C = 5 × 7 times N. M;
3	Ax. 1. I.	$\therefore$ A . D = B . C.

CASE II. Let A & B be *incommensurable,* and $\therefore$ also C and D.

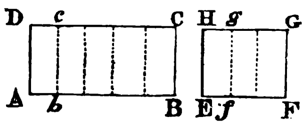
D. 1	H.	$\because$ A : B = C : D;
2	Pr M, V.	and P, Q may approach nearer A, C than any assigned difference,
3		P and Q also containing like parts of B and D;
4		$\therefore$ P . D = Q . B.
5	Pr. M, V.	And $\because$ there may be taken like parts of B, D continually less and less;
6		and $\because$ P and Q increase towards A and D within any assigned difference;

7		and also ∵ P . D and Q . B, by increasing together, approach nearer to A . D and C . D than any assigned difference;
8	Theor. I.	∴ A . D = C . B, or B . C.
9	Rec.	Therefore, *if four st. lines be proportionals, &c.* Q. E. D.

USE & APP. I. The Theory of Limits may be applied to establish Prop. 1, VI., both for bases that are commensurable, and for those which are incommensurable; thus,

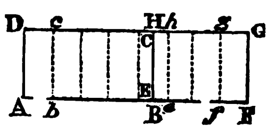
The Rectangles AC *and* EG, *having the same alt.* AD = EH, *are to each other as their bases,* AB *and* EF.

CASE I. Let AB, EF be *commensurable, i. e.,* exact multiples of Ab = Ef.



C. 1	*Sup.*	Let AB = 5 Ab, and EF = 3 Ef;
2	11, I.	and from each . of the divisions of the bases erect perpendiculars, making 5 eq. ▭s on AB, and 3 eq. ▭s on EF.
D. 1	C. 2.	∵ ▭ AC = 5 Db, & ▭ EG = 3 Hf;
2		∴ ▭ AC : ▭ EG = 5 : 3;
3	C. 1.	and ∵ AB = 5 Ab and EF = 3 Ef;
4		∴ ▭ AC : ▭ EG = AB : EF.
5	*Sim.*	In like manner, whatever may be the ratio.

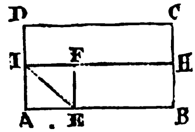
CASE II. Let the bases, AB, EF, be *incommensurable,* i. e., not exact multiples of Ab and Ef.



C. 1	Sup.	Let AB, EF form one st. line, AF = 8 Ab;
2		and let e be the · in the division nearest to B or E.
D. 1	C. 1 & 2	∵ AF and Ae are commensurable, AF = 8 Ab & Ae = 5 Ab;
2		∴ AehD : AFGD = Ae : AF;
3		*or,* $\frac{\text{AEHD}}{\text{AFGD}} + \frac{\text{E}eh\text{H}}{\text{AFGD}} = \frac{\text{AE}}{\text{AF}} + \frac{\text{E}e}{\text{AF}}$.
4		But E$e$$h$H and E$e$ may be reduced as small as we please, while AFGD and AF remain unchanged;
5	Sch. 4. 16, VI.	∴ by the theory of limits, AEHD : AFGD = AE : AF.

II. *The Rectangles* AC *and* AF *are to each other as the product of their bases by their altitudes;* i. e., as AB . AD : AE . AI.

C.	Pst. 2, I.	Produce the line IF to cut BC in H.
D. 1	Last Th.	$\frac{ABHI}{AEFI} = \frac{AB}{AE}$ & $\frac{ABCD}{ABHI} = \frac{AD}{AI}$;
2	× the equat.	and $\frac{ABHI \times ABCD}{AEFI \times ABHI} = \frac{AB \times AD}{AE \times AI}$;
		or, $\frac{ABCD}{AEFI} = \frac{AB\,.\,AD}{AE\,.\,AI}$.
3	*Sup.*	Let AI = AE = *unity;* then □ AEFI is a *unit of surface.*
4	Bk. II. p. 145.	∴ Units of surface in ▭ AC = AB . AD; *i. e.* = the linear units in AB × the linear units in AD.



N. B. In this sense the *Area of a* ▱ = product of base and perpendicular.

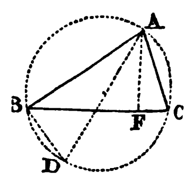
And ∵ *a* △ = ½ ▱ on the same base and of the same altitude;

∴ *Area of a* △ = ½ the product of the base and perpendicular.

III. Among other Theorems the following is easily deducible;

The rect. contained by AB, AC, *any two sides of a* △, ABC, *is equal to the rect. contained by* AF, *its alt., or perp. to the third side,* BC, *from the opp.* ∠ BAC, *and by the diam.* AD, *of the circumscribing* ⊙ BACD.

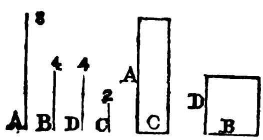
D. 1	31, III, & H.	∵ ∠ ABD is a rt. ∠ and also ∠ AFC;
2	21, III.	and ∠ ADB = ∠ ACF;
3	Cor. 33, 32, I.	∴ △s ADB and ACF are equiangular;
4	4, VI. 16, VI.	and ∴ AB : AD = AF : AC; and AB . AC = AD . AF.
5	Rec.	Therefore, the rect. contained by any two sides, &c. Q. E. D.



PROP. 17.—THEOR.

COR. to Prop. 16. *If three st. lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean*; and conversely, *if the rectangle contained by the extremes be equal to the square of the mean, the three st. lines are proportionals.*

CON. 3, I. DEM. 7, V. 16, VI.



E. 1	Hyp.	CASE I. Let A : B = B : C;
2	Conc.	then A . C = B . B, or B^2.
C.	3, I.	Take D = B
D. 1	H & 7, V.	∵ A : B = B : C, & B = D; ∴ A : B = D : C;
2	16, VI.	& ∴ ▭ A . C = ▭ B . D.
3	C. Conc.	but ∵ B = D ∴ ▭ B . D = B^2; & ∴ ▭ A . C = B^2.

E 1.	Hyp.	CASE II. And if ▭ A . C = B^2
2	Conc.	then A : B = B : C.
D. 1	H.	∴ A . C = B^2, & B^2 = B . D; ∴ A . C = B . D;
2	16, V.	& ∴ A : B = D : C;
3	C. Conc.	but B = D; ∴ A : B = B : C.
4	Rec.	Therefore, *If three st. lines be proportionals &c.*

Q. E. D.

Alg. & Arith. Hyp. Let a 2 : b 4 = b 4 : c 8.

Alg. If $\frac{a}{b} = \frac{b}{c}$;

$\times (b \times c)$ $\frac{abc}{b} = \frac{bbc}{c}$, i. e. $ac = b^2$;

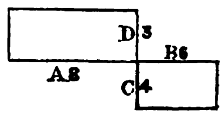
Conversely, ∵ $ac = b^2$;

$\div (b \times c)$ ∴ $\frac{ac}{bc} = \frac{b^2}{bc}$, i. e. $\frac{a}{b} = \frac{b}{c}$.

Arith.	If $\frac{2}{4} = \frac{4}{8}$;	
$\times (4 \times 8)$	$\frac{2 \times 4 \times 8}{4} = \frac{4 \times 4 \times 8}{8}$; i. e., $2 \times 8 = 4 \times 4$.	
Conv.	$\because 2 \times 8, = 4 \times 4$;	
$\div (4 \times 8)$	$\therefore \frac{2 \times 8}{4 \times 8} = \frac{4 \times 4}{4 \times 8}$; i. e., $\frac{2}{4} = \frac{4}{8}$.	

USE & APP. I. *The Demonstration of Proportion in Arithmetic*, commonly named "*The Rule of Three,*" depends on the last four Props. *viz.*, 14, 15, 16 & 17, bk. VI.; and from that demonstration we deduce the Rule for *finding a fourth Proportional*, D, *to three given terms*, A 8, B 6, and C 4.

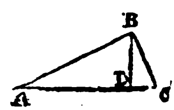
E.	Hyp.	Suppose the 4th propl. D to be found.
D. 1	16, VI.	$\because$ A.D = B . C; $\therefore$ D = $\frac{\text{B . C}}{\text{A}}$



Deduction,	1.	$\frac{B \,.\, C}{A} = \frac{6 \times 4}{8} = \frac{24}{8} = 3 = D.$
"	2.	$\frac{B \,.\, C}{D.} = \frac{6 \times 4}{3} = \frac{24}{3} = 8 = A.$
"	3.	$\frac{A \,.\, D}{B} = \frac{8 \times 3}{6} = \frac{24}{6} = 4 = C.$
"	4.	$\frac{A \,.\, D}{C} = \frac{8 \times 3}{4} = \frac{24}{4} = 6 = B.$

II. The Properties of Equiangular triangles established in Prop. 4 and 5, and those of Proportionals in Prop. 16 & 17, lead to a very clear demonstration of 47, I; *that in a rt.* $\angle$ *d* $\triangle$ ABC, *the square of the hypotenuse* AC, equals *the sum of the squares of the other two sides,* AB & BC.

C.	12, I.	From the vert. of the rt. $\angle$ B, draw BD $\perp$ AC.
D. 1	8, VI.	$\because$ $\triangle$ ADB is eq. ang. to $\triangle$ ABC which is eq. ang. to $\triangle$ BDC.
2	4, VI.	$\therefore$ AC : AB = AB : AD, and AC : BC = BC : DC;
3	17, VI.	$\therefore$ AB^2 = AC . AD; and BC^2 = AC . DC.
4	Ax. 2, I.	+ *equals*; $\therefore$ $AB^2 + BC^2$ = AC . AD + AC . DC.



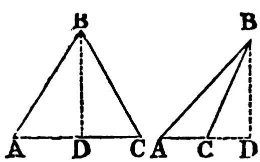
D. 5	C. 1, II.	Now ∵ side AC is com.
		∴ $AB^2 + BC^2 = AC.(AD + DC)$;
6	Conc.	but $AD + DC = AC$;
		∴ $AB^2 + BC^2 = AC.AC = AC^2$.
7	Rec.	∴ *In a rt. ∠d triangle, &c.* Q. E. D.

Thus the MOST IMPORTANT PROPERTIES of Plane Geometry are reduced to the *single property*; that "in similar triangles the sides about the equal angles are proportional.

III. Some of the deductions, made in connection with this demonstration, may be stated *in other words*, yet still in accordance with Prop. 16 & 17 ; thus, instead of the inference, "*if a perp. be drawn from the rt. ∠ to the hypotenuse the square of either side is equal to the rect. under the hypotenuse and seg. adj. to that side*, we say, "either side, as AB, is a mean proportional between the whole hypotenuse AC, and the seg. AD adj. to that side."

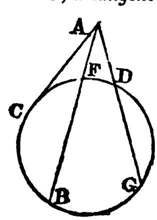
Also, instead of "*the square of the perp. BD, on the hyp. AC, will equal the rect, under the segs. of the hyp. AD . DC,*" we declare,—"the perp. BD, is a mean propl. between the segs. of the hyp. AD & DC."

And in place of the deduction, "*in every △ if a perp. BD be drawn from the vertex B, to the base AC,* fig. 1, *or base produced AD,* fig. 2, *the difference of the squares of the sides BA, BC, shall be equal to the difference of the squares of the segs of base, AD, CD, or of the base produced,*" we substitute, "the base is to the sum of the sides as the difference of the sides is to the difference of the segments of the base, or sum of the segments of the base produced.



IV. As instances in which the use of Props. 16 & 17 very considerably shortens the demonstrations of other propositions, let us take ;

Ex. 1. *If from a point A, there be drawn two st. lines one AC, a tangent to a circle, the other, AB, a secant, then the tang. will be a mean propl. between the whole secant AB, and its ext. segment, AF.*



D. 1	H. & 36, III.	∵ AC touches and AB cuts the ⊙,
		∴ $AC^2 = AB.AF$;
2	17, VI.	∴ BA : AC = AC : AF ; *i. e.*, the tang. is a mean propl. to AB & AF.

Ex. 2. ***If from a . A there be drawn several st. lines AB, AG cutting the circle, then the whole secants, AB, AG, will be one to another* inversely *as their external segments., AF, AD.***

D.	1	36, III.	∵ $AC^2 = BA . AF$ & also $= GA . AD$;
	2	Ax. 1, I.	∴ $BA . AF = GA . AD$;
	3	16, VI.	& ∴ $BA : GA = AD : AF$.

N.B. For additional Examples of the Use and Application of Propositions 16 & 17, the long series may be consulted given in LARDNER's *Euclid* pp. 204—208.

PROP. 18.—PROB. *(Important.)*

Upon a given st. line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

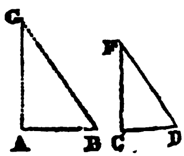
CON, 23, I. At a given . in a given st. line to make a rectl. ∠ equal to a given rectl. ∠.

32, I. If a side of a △ be prod. the ext. △ = two int. & opp. ∠s; and the three int. ∠s of every △ together = two rt. ∠s.

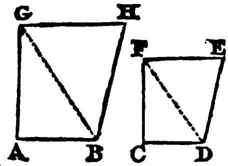
Cor. 3, 32, I. In △s if two ∠s in each be eq. the third ∠s also eq.

DEM. 4, VI. 22, V. If there be any no. of Ms, and as many others, which taken two and two in order have the same R; the first shall have to the last of the first Ms, the same R, which the first has to the last of the others. Def. 1, VI. Ax. 2, 3, I.

Def. 12, V, Homologous sides.

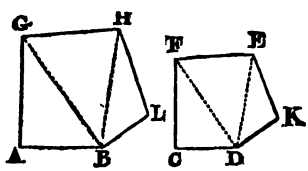


E.	1	Dat.	CASE I. Given AB and △CDF;
	2	Quæs.	on AB to desc. a △ sim. to △CDF, & AB homogous to CD.
C.	1	23, 1.	At A make ∠A = ∠C, & at B, ∠B, = ∠D;
	2	Cor. 3, 32, I.	then rem. ∠G = rem. ∠F, & △ABG is eq. ang. to △DF.
D.	1	C. 2, 4, VI.	∵ △ABG is eq. ang. to △CDF, ∴ those △s are similar;
	2	Def. 12, V.	∴ BA : AG = CD : CF; & AB, CD are homolog. sides.
	3	Hence	Hence on the given line AB &c.



E. 1	Dat.	Case II. Given AB & rectl fig. CDEF of four sides;
2	Quæs.	on AB to desc. a rectl fig. similar, and similarly situated to fig. CDEF.
C. 1	Pst. 1. 23, I.	Join DF; at A make $\angle A = \angle C$, and at B, $\angle ABG = \angle CDF$;
2	23, I. Ax. 3.	then rem. $\angle AGB =$ rem $\angle CFD$; and $\triangle FCD$ is eq. ang. to $\triangle GAB$.
3	32, I.	Again at G, make $\angle BGH = \angle DFE$; & at B, $\angle GBH = \angle FDE$;
4	32, I. Ax. 3.	then rem. $\angle H =$ rem. $\angle E$, & $\triangle GBH$ is eq. ang. to $\triangle FDE$.
D. 1	C. 2 & 3.	$\because \angle AGB = \angle CFD$, & $\angle BGH = \angle DFE$;
2	Ax. 2.	$\therefore$ the whole $\angle AGH =$ the whole $\angle CFE$.
3	*Sim.*	Thus $\angle ABH = \angle CDE$, $\angle A = \angle C$, & $\angle H = \angle E$:
4	Cor. 3, 32, I.	$\therefore$ rectl. fig. ABHG is eq. ang. to rectl. fig. CDEF.
5	D. 3. 4, VI.	Also $\therefore \triangle GAB$ is eq. ang. to $\triangle FCD$, $\therefore BA : AG = DC : CF$;
6	C. 4.	& $\because \triangle BGH$ is eq. ang. to $\triangle DFE$;
7	4, VI.	$\therefore AG : GB = CF : FD$; & $GB : GH = FD : FE$;
8	22, V.	$\therefore$ *ex æq.* $AG : GH = CF : FE$.
9	*Sim.* 4, VI.	So $AB : BH = CD : DE$, & $GH : HB = FE : ED$.
10	Rec.	Now, $\because$ rectl. fig. on AB. is eq. ang. with the rectl. fig. on CD, and their sides proportional;
11	Def. 1, VI.	$\therefore$ figures, ABHG & CDEF are similar.

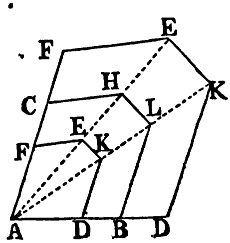
E. 1	Dat.	Case III.—Given AB and rectl. fig. CDKE F of five sides;
2	Quæs.	on AB to desc. a rectl. fig. similar, and similarly situated to fig. CDKEF.



C. 1	Pst. 1, I, & Case II.	Join DE, and on AB desc. rectl. fig. ABHG sim- and similarly situated with fig. CDEF;
2	23, I.	at B made ∠ HBL = EDK, and at H, ∠BHL = ∠ DEK;
3	32, I. Ax. 3	then rem. ∠ L = rem. ∠ K.
D. 1	C. Def. 1.VI	∵ fig. AH is sim. to fig. CE, ∴ ∠ GHB = ∠ FED;
2	C. 2.	and ∵ ∠ BHL = ∠ DEK, ∴ whole ∠ GHL = whole ∠ FEK.
3	*Sim.*	So ∠ ABL = ∠ CDK; ∴ pent. GL is eq. ang. with pent. FK.
4	C. Def. 1,VI	And ∵ fig. AH is sim. to fig. CE; ∴ GH : HB = FE : ED;
5	4, VI. 22, V.	but HB : HL = ED : EK; ∴ *ex. æq.* GH : HL = FE : EK
6	*Sim.*	So AB : BL = CD : DK.
7	C. 2, 4, VI.	And ∵ △ BLH is eq. ang. with △ DKE; ∴ BL : LH = DK : KE.
8	D. 3.	∵ pent. GL is eq. ang. with pent. FK, and their sides proportional.
9	Conc.	∴ pent. GL is similar to pent. FK.
10	*Sim.*	By a like process a *hexagon* &c., may be described on AB.
11	Rec.	Therefore, *Upon a given st. line, &c.* Q. E. F.

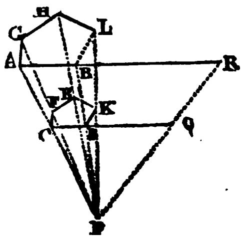
Sch. I. A more simple way *for making a rectil. fig.*, as ADKEF, *similar to a rectil. fig.*, ABLHC, *would be;*

C. 1	3, I.	*Method* 1. On AB or AB produced set AD:
2	Pst. 1, III.	From the centre of similarity A, divide fig. ABLHC, into △s, by indefinite lines AK, AE;
3	31, I.	make DK ∥ BL, KE ∥ LH, and EF ∥ HC;
4	Sol.	then fig. ADKEF is similar to fig. ABLHC.
D. 1	C. 3.	∵ the respective lines are parallel;
2	29, I.	∴ ∠s at D = ∠s at B; ∠s at K = ∠s at L, &c. and ∠s at A com. *i. e.*, the respective △s are eq. ang.;
3	4, VI.	∴ the sides about the eq. ∠s are propl.
4		∴ AK : AL = DK : BL, and AK ; AL = KE : LH;



	5		∴ DK : BL = KE : LH.
	6	*Sim.*	So KE : LH = EF : HC, and so on;
	7	Conc.	∴ the sides about the eq. ∠s are proportional and the figures similar. Q. E. F.

Method 2. On AB to place a fig. similar to the rectl. fig. CDKEF.



C.	1	31, I.	At a convenient distance from CD place AB ∥ CD;
	2	Pst. 1, I.	join AC and BD, if AB ; > CD, AC and BD will meet on being produced; let them meet in the . P;
	3	Psts. 1, 2, I.	join PF, PE, PK, and produce them indef;
	4	31, I.	through . A draw AG ∥ CF, through . G, GH ∥ FE, and through . H, HL ∥ EK; and join BL;
	5	Conc.	then fig. AL on AB is sim. to fig. CK on CD.
D.	1	C. 1, 4.	∵ the respective lines CF, AG, FE and GH, EK and HL, &c. are parallel;
	2	29, I.	∴ the ∠s about the resp. points C & A, F & G, E & H &c. are equal;
	3	4, VI.	& ∴ the resp, △s are equiangular and similar;
	4		∴ CD : DP = AB : BP : & DP : DK = BP : BL;
	5	22, V.	∴ *ex æq.* CD : DK = AB : BL.
	6	*Sim.*	So the sides about the other eq. ∠s are proportional.
	7	Conc.	∴ fig. AL on AB is similar to fig CK on CD. Q.E.D.

N.B.—Should AB = CD the demonstration must be derived trom 32, I.

II. *To construct a rectl. fig.*, as in the last diagram, *similar to a given rectl. fig. AL on AB, and having its perimeter equal to a given st. line CQ.*

C.	1	Pst. 2, I. 3, I.	Prod. AB until AR = the perimeter of the given fig. AL
	2	12, VI.	to AR, CQ & AB take a 4th propl. CD;
	3	3, I.	set the 4th propl CD on CQ;
	4	18, VI.	and on CD desc. a rectl fig. CK sim. to fig. AL;
	5	Conc.	then the perimeter of fig. CK = the st. line CQ.
D.	1	C. 2	Now per. AR : per. CQ = AB : CD;
	2	C. 1, & H.	but per. of fig. AL = AR, and per. of fig. CK = CQ;
	3	Conc.	∴ a fig. has been drawn, similar to AL & with perimeter = CQ.

III. "As many figures of the same species with different areas can be constructed on the same right line as the figure of the proposed species has sides of different lengths."

IV. Anticipating the next two propositions, and using the same construction as in the last figure but one, we may also prove, that *Similar triangles and polygons are to one another as the squares of their homologous sides.*

$$\text{For, } \frac{\text{Area AEF}}{\text{Area AHC}} = \frac{AE^2}{AH^2} = \frac{AK^2}{AL^2} = \frac{AD^2}{AB^2}, \text{ \&c.}$$

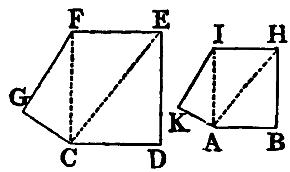
$$\therefore \text{Area AEF} : AD^2 = \text{Area AHC} : AB^2.$$

$$\text{In like manner, } \frac{\text{Area AKE}}{AD^2} = \frac{\text{area ALH}}{AB^2}; \text{ and } \frac{\text{area ADK}}{AD^2} = \frac{\text{area ABL}}{AB^2}$$

$$\textit{Adding equals, } \text{Area ADKEF} : AD^2 = \text{area ABLHC} : AB^2;$$

$$\therefore \text{Area ADKEF} : \text{area ABLHC} = AD^2 : AB^2.$$

Use & App. Nearly all *the practical methods of taking a Plan of* any building or place, or of drawing a Map of a field, of an estate, or of a whole country, are dependent on this Proposition. The lines made use of in the Plan or Map are *proportionals* to the existing lines in a building, estate, or country, and *Representatives* of the same values.



The object of the Surveyor is to lay down a figure, as ABHIK, exactly like in shape to the *Prototype*, or original, as CDEFG, possessing similar angles and proportional sides; and this he accomplishes by actual measurements or calculations of the sides and angles, and then by reducing the lengths of the lines in his drawing or plan in exact proportion to the original Features or Outlines. The very same angles are given in the Map that exist in the field or the country that is surveyed; thus the figure ABHIK is a reduced copy of the fig. CDEFG, preserving the $\angle$s at A, B, &c., equal to those at C, D, &c.,—but giving the lines AB, BH, &c. in proportion only to the lines CD, DE, &c.

Whenever we have to make a Design, or a Model, we use this Proposition either actually or virtually, so that its application extends to all the Problems of Geodœsia, or the Art of Measuring and representing surfaces,—to Plans, Maps, and Geometrical Drawings of every kind.

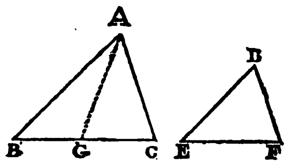
Prop. 19.—Theor. (*Very important.*)

Similar triangles are to one another in the duplicate ratio of their homologous sides; i. e., as the squares of their like sides.

Con. 11, VI. Dem. Def. 12, V.—16, V. If four Ms of *of the same kind* be propls., they shall also be propls. when taken alternately.

11, V. 15, VI. Def. 10, V. When three Ms are propls. the 1st is said to have to the 3rd the duplicate R. of that which it has to the 2nd. 1, VI.

E. 1	Hyp. I.	Let $\triangle ABC$ be sim. to $\triangle DEF$, & $\angle B = \angle E$;
2	,, 2.	& let $AB : BC = DE : EF$,
	Def. 12, V.	the side BC being homol. to EF;
3	Conc.	then $\triangle ABC : \triangle DEF = BC^2 : EF^2$.



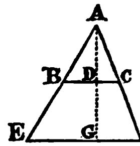
C. 1	11, VI.	Take BG a 3rd propl. to BC & EF, *i.e.*, $BC : EF = EF : BG$; and join GA.
D. 1	H. 2. 16, V.	$\because AB : BC = DE : EF$; $\therefore$ *altern.* $AB : DE = BC : EF$;
2	C. 11, V.	but $BC : EF = EF : BG$; $\therefore AB : DE = EF : BG$;
3	Sc. 1, 16, VI.	$\therefore$ the sides about the eq. $\angle$s. are recip. propl.,
4	15, VI.	$\therefore \triangle ABG = \triangle DEF$.
5	C.	And $\because BC : EF = EF : BG$;
6	Def. 10, V.	$\therefore BC : BG$ has the duplicate R of $BC : EF$;
7	1, VI.	but $BC : EF$ or $BG = \triangle ABC : \triangle ABG$;
8	D. 6, 7.	$\therefore \triangle ABC : \triangle ABG = BC^2 : EF^2$;
9	D. 4.	but $\triangle ABG = \triangle DEF$; $\therefore \triangle ABC : \triangle DEF = BC^2 : EF^2$.
10	Rec.	Therefore, *Similar triangles are &c.* Q. E. D.

Cor. *If three st. lines are proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.*

Sch I. The duplicate ratio of two st. lines is the same with the ratio of their squares; thus similar triangles are to one another as the squares of their homologous sides.

II. This Proposition might be deduced as a particular case of Prop. 1, VI, by the 2nd Cor. "that triangles are to one another in the ratio compounded of the ratios of their bases and altitudes;" for, when triangles are similar, their altitudes are in the proportion of their bases, and by Def. 12, V. this compound ratio is the duplicate ratio of their bases.

USE and APPL. I. The 19th Prop. supplies a *general method of reasoning* concerning all similar geometrical figures; for we declare from it *that the areas of similar triangles are to one another as the squares for the corresponding sides.*



E. 1	Hyp. 1.	Let $\triangle$s ABC, AEF be similar;
2	„ 2	& AD $\perp$ BC; and AD produced, or AG, $\perp$ EF;
3	Cor. 3, 32, I.	& $\triangle$s ADB, AGE equiangular, then Area of $\triangle$ABC : Area of $\triangle$AEF $= BC^2 : EF^2$, or $AB^2 : AE^2$;
D. 1	4, VI.	$\because$ AD : AG = BC : EF; & BC : EF = BC : EF;
2		on multiplying these equals together. AD . BC : AG . EF $= BC^2 : EF^2$.
3	Use 2, 16, VI.	But AD . BC = Area of ▱ on BC with alt. AD;
4		and AG . EF = Area of ▱ on EF „ AG.
5	Use 2, 16, VI.	And Area of $\triangle$s = ½ Area of ▱s on the same base and altitude,
6	Sch. 1. 19, VI.	$\therefore$ Area of $\triangle$ ABC : Area of $\triangle$ AEF $= BC^2 : EF^2$.
7	*Sim.*	Thus are the Areas as the squares of any other pair of corresponding sides; for Area of $\triangle$ ABC : Area of $\triangle$ AEF $= AB^2 : AE^2$.

Ex. 1. From a given $\triangle$ AEF to cut off a part ABC = ¼ of $\triangle$AEF in area, by a line BC ∥ EF.

Since, Area $\triangle$AEF : Area $\triangle$ABC $= EF^2 : BC^2$; *or* $1 : \frac{1}{4} = EF^2 : BC^2$,

$$\therefore BC^2 = \frac{EF^2}{4}, \textit{ or } BC = \frac{EF}{2}.$$

Ex. 2. If the Area of a triangle be 36000 sq. feet, of which the base is 900 feet, what will be the area of a similar triangle of which the base is 450 feet.

Take x as representative of the unknown area;

$\because 36000 : x = 900^2 : 450^2$; $\therefore 810000\,x = 7290000000$;

and $x = 7290000000 \div 810000 = 9000$ sq. feet.

Ex. 3. The base EF, of a triangle is 3240 links, and its perpendicular 840 links; required the distance, from the base, of the parallel line, BC, which shall divide the triangle into two equal parts.

Here, Area of $\triangle$ AEF $= \frac{3240 \times 840}{2} = 2721600$ sq. links;

and Area of $\triangle$ ABC $= \frac{2721600}{2} = 1360800$ " "

Now AG$-$AD $=$ DG distance of parallels BC & EF;

To find AD; Area $\triangle$ AEF : Area $\triangle$ ABC $=$ AG^2 : AD^2,

Then $AD^2 = \frac{1360800 \times 840^2}{2721600} = 352800$;

$\therefore$ AD $= \sqrt{352800} = 593.978$ links.

And $840 - 593.978 = 246.022 =$ DG, distance of parallel BC from EF.

II. When in similar triangles, any side of the one, as AE, is double that of the other, the area of the one triangle is four times that of the other; for the Areas are as the squares of the sides; and $2^2 : 1^2 = 4 : 1$.

Prop. 20.—Theor.

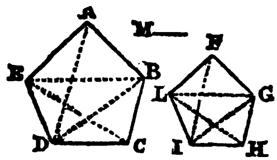
Similar Polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have, and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Similar polygons are divisible into similar triangles equal in number and homologous to the whole polygons; and the polygon has to the polygon the duplicate ratio which the homologous side has to the homologous side."—Euclid.

Con, Pst. 1. I. Dem. Def. 1, VI. 6, VI. 4, VI. 32, I. Ax. 3, I. 22, V. 11 V. 12, V. If any number of Ms be propls, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents. 19, VI.

E. 1	Hyp. 1.	Let pol. AD be similar to pol. FI, and the $\angle$s at A, B, C, D, E respectively $=$ $\angle$s at F, G, H, I, L;
2	" 2.	and let the sides AB, BC, CD, DE, EA be respectively homol. to the sides FG, GH, HI, IL, LF;
3	Conc. 1.	then pols. AD and FI, are divisible into $\triangle$s of which each contains the same number of sim. $\triangle$s;

4	,, 2.	of these sim. △s each has to each the same ratio as the pol. AD to the pol. FI.
5	,, 3.	and pol. AD : pol FI has the duplicate ratio of the R, AB : FG.



C.	Pst. 1, I.	Join E with the ∠ . s B, C; and L with the ∠ . s G,H.

CASE I. The pols. AD & FI are divisible into ∠s, &c.

D. 1	H. 1, Def. 1, VI.	∵ pol. AD is sim. to pol FI ∴ ∠BAE= ∠GFL, & BA : AE = GF : FL;
2	H. 1, 2.	And ∵ △s ABE, FGL have ∠A = ∠F, and their sides proportionals;
3	6, VI. 4, VI.	∴ △ABE is eq. ang. & sim. to △FGL,
4		∴ ∠ABE = ∠FGL.
5	Hyp. Def. 1, VI.	And ∵ pol. AD is sim. to pol. FI, ∴ whole ∠ABC = whole ∠FGH;
6	32, I. Ax. 3.	& ∴ rem. ∠EBC = rem. ∠LGH.
7	D. 3. 4, VI.	And ∵ △ABE is sim to △FGL, ∴ EB : BA = LG : GF;
8	Def. 1, VI.	Also ∵ of sim. pols. ∴ AB: BC=FG: GH;
9	22, V.	∴ *ex. æq.* EB : BC = LG ; GH.
10	D 9.	And ∵ the sides about eq. ∠s are propls,
11	6, VI. 4, V1.	∴ △EBC is eq. ang. & sim to △LGH.
12	*Sim.*	So △ECD is eq. ang. and sim to △LHI.
13	Conc.	∴ Sim. pols. AD, FI are divided into the same number of sim. △s.

CASE II. Also $\left.\begin{array}{l}\triangle ABE : \triangle FGL \\ \triangle EBC : \triangle LGH \\ \triangle ECD : \triangle IHL\end{array}\right\} =$ pol. AD : pol. FI.

Case III. And pol. AD : pol. FI has the duplicate R of AB : FG.

D. 1	CaseI.19,VI	∴ △ABE is sim. to △FGL, ∴ △ABE : △FGL = BE^2 ; GL^2
2	*Sim.*	So △BEC : △GLH = = $BE^2 : GL^2$.
3	11, V.	∴ △ ABE : △ FGL = △ BEC : △ GLH.
4	CaseI.19,VI	Again ∵ △ EBC is sim. to △ LGH, ∴ △ EBC : △ LGH = $EC^2 : LH^2$;
5	*Sim.*	So △ ECD : △ LHI = $EC^2 : LH^2$,
6	11, V.	∴ △ EBC : △ LGH = △ ECD : △ LHI;
7	D. 3.	but △ EBC : △ LGH = △ ABE : △ FGL,
8		∴ △s ABE : FGL = EBC : LGH = ECD : LHI;
9	12, V.	∴ antec. : conseq. = all the antecs. : all the conseqs. *i. e.* △ ABE : △ FGL = pol. AD : pol. FI.
10	19, VI.	But △ ABE : △ FGL = $AB^2 : FG^2$,
11		∴ pol. AD : pol. FI = $AB^2 : FG^2$.
12	Rec.	Wherefore, *Similar Polygons &c.* Q. E. D.

Cor. I. *Similar figures of four sides, or of any number of sides,* as already proved of triangles 19, VI.; *are to one another in the duplicate ratio of their homologous sides.*

Cor. II. *If three lines be proportionals, then,* as was proved for triangles, Cor. 19, VI., *the first shall be to the third as any polygon on the first is to the similar, and similarly described polygon on the second.*

C.	11, VI.	To AB, FG two homol. sides take M, a third proportional.
D. 1	Def, 10. V.	Then AB : M has the dupl. R of AB : FG;
2	Cor. 1,20, VI.	but pol. AD : pol. FI = $AB^2 : FG^2$;
3	11, V.	∴ AB : M = fig. on AB : fig. on FG.
4	Rec.	∴ *If three lines be proportionals, &c.* Q.E.D.

COR. III. Because all squares are similar figures, *the ratio of any two squares to one another is the same with the duplicate ratio of the sides;* and hence also, *any two similar rectilineal figures are to one another as the squares of the homologous sides.*

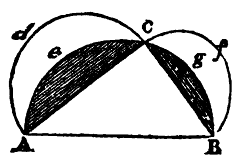
COR. IV. *In similar figures, their perimeters are to one another as the ratio of the homologous sides.*

∵ AB,BC &c. : FG,GH &c. = AB : FG =BC : GH;
∴ AB + BC + CD &c. : FG + GH + HI &c. = AB : FG.
i. e., Perim. of fig. AD ; perim. of fig. FI = AB : FG.

COR. V. The homologous diagonals being sides of similar triangles are also homologous sides; and therefore, *perimeters of similar figures are as their homologous diagonals.*

COR. VI. A Circle and its inscribed Polygon of an infinite number of sides do not differ from each other in any degree however small, but that a smaller difference might be assigned; and *practically*, they may be considered of identical values. As therefore it has been predicated of similar polygons, 20, VI, that "they are to each other as the squares of their corresponding sides," whether diagonals or sides;—so it may be predicated of *all circles*, that being similar figures, they *are to each other as the squares of their respective diameters, radii, and circumferences.*

COR. VII. *If on the three sides*, AB, AC, CB, *of a rt. ∠d triangle* ACB, *similar figures be described* as semicircles, A *e* C *g* B, A *d* C and C *f* B; *the figure on the hypotenuse*, AB, *will be equal to the sum of the similar figures on the two other sides*, AC, CB.



D. 1	Cor. 6, 20, VI.	∵ Semicircles, or any sim. fig., on the three sides, are as the squares on those three sides;
2	47, I.	& ∵ $AB^2 = AC^2 + CB^2$,
3		∴ Semicircle on AB = Semic. on AC + Semic. on CB.

HIPPOCRATES, of Chios, a Pythagorean philosopher, who lived about 460 B.C., is said to have made this application of the universal principle, which under different forms appears in 47, I., and 31, VI., that "if *three similar figures be described upon the sides of a rt. angled triangle, the contents of that which is described upon the hypotenuse will be equal to the sum of the contents of the figures described upon the sides.*"

Hence, was deduced,

COR. VIII. That *the Lunes* $AeCd$, $CgBf$, formed by describing semicircles, as $AeCgB$, AdC & CfB, on the sides of a rt. ∠d triangle, *are equal in area to the right angled triangle* ACB.

D. 1	Cor. 7. 20, VI.	Semic. on AB = semic. on AC + semic. on CB;
2	*Sum.*	from semic. on CB take away the shaded segments AeC & CgB;
3		The remainder is the rt. ∠d △ ACB;
4	*Sum.*	From semicircles on AB, CB take away the same segments;
5		The remainders are the two lunes $AeCd$, $BgCf$;
6	Ax. 4, VI.	∴ the Lunes $AeCd + BgCf = \triangle$ ACB.

N. B. This is the first known instance in which a curvilinear space was reduced to an equivalent rectilinear space.

SCH. 1. Further elucidation of Prop. 20, will be found in LARDNER'S Euclid, pp. 210—214, and in pp. 127, 134 of this work.

2. In circles, which are all similar, when the segments are similar, the radii and the chords become homologous sides.

USE & APP. I. By this proposition, *a rt. lined figure may be increased or diminished in any ratio*

Thus, to make a pentagon *five* times the size of the pent. on DC, (in last figure but one.);

Take a side AE = 2; and between AE and 5 times AE, *i. e.*, 10, find the mean proportional;—it is $\sqrt{2 \times 10} = \sqrt{20} = 4.472136$;

A sim. fig. on a side of 4.472136 will be five times the pent. on AE.

II. When the homologous sides are known *the Proportion of one figure to another* will be obtained by finding a third proportional to any two corresponding sides.

III. The *increase or diminution of Circles* is effected in the same way.

Thus one circle is on a diam. of 1,—another is to be constructed 4 times larger; required the diam. of the second circle.

The mean propl. to 1 & 4 is $\sqrt{1 \times 4} = \sqrt{4} = 2$.

$\therefore$ the diam. is 2 for the required circle.

Or, Given the diameters 1 & 2, required the magnitude of the second circle when compared with the first.

Here, $\frac{2 \times 2}{2} = \frac{4}{1} = 4$,—the second circle is four times larger than the first.

IV. In two similar figures, if *of the areas and corresponding sides, any three be given, the fourth may readily be found.* The principle employed is, that the areas of similar figures are to one another as the squares of their homologous sides, and *vice versâ.*

For, (fig. 20, VI.) $\triangle\ ABE : \triangle\ FGL = AB^2 : FG^2$;
$\triangle\ BCE : \triangle\ GHL = BC^2 : GH^2$;
and $\triangle\ CDE : \triangle\ HIL = CD^2 : HI^2$.

Hence, $\triangle$s $ABE + BCE + CDE : \triangle$s $FGL + GHL + HIL = AB^2 : FG^2$.

i.e. fig. $ABCDE$: fig. $FGHIL = AB^2 : FG^2$;

$$\text{Thus } FG^2 = \frac{FGHIL \times AB^2}{ABCDE};\quad AB^2 = \frac{ABCDE \times FG^2}{FGIHL};$$

$$\text{Area } ABCDE = \frac{FGHIL \times AB^2}{FG^2};$$

$$\text{and Area } FGHIL = \frac{ABCDE \times FG^2}{AB^2}.$$

Ex. 1. Two similar hexagons are respectively of the Areas of 2500 sq. yards and 3000 sq. yards; a side of the first is 5 lineal yards; required the length of the corresponding side of the other hexagon.

Here, the unknown side $x = \sqrt{\frac{3600 \times 5 \times 5}{2500}} = \sqrt{36} = 6$ lineal yards.

Ex. 2. One of the sides of the base of a pyramid measures 50 yards, and the area of the base 7500 sq. yards;—in a similar pyramid, what is the area of the base when the corresponding side is 60 yards?

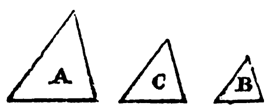
The Area $x = \frac{7500 \times 60 \times 60}{50 \times 50} = \frac{270000}{25} = 10800$ sq. yards.

Prop. 21.—Theor.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Dem. Def 1, VI. Ax. 1, I. 11, V.

E. 1	Hyp.	Let figures A & B be each sim. to fig. C.
2	Conc.	then fig. A is sim. to fig. B,
D. 1	H.Def.1,VI.	∵ A is sim. to C, ∴ A is eq. ang. with C, & their homol. sides are proportionals.
2	H. Def. 1, VI	∵ B is sim. to C, ∴ B is eq. ang. with C, and their homol. sides. propls.
3	Ax. 1, I. 11, V.	∴ A & B are each eq. ang. with C, and their homol. sides propls.
4	Def. 1, VI.	∴ rectl. fig. A is sim. to rectl. fig. B.
5	Rec.	Therefore, *Rectilineal figures which are similar, &c.* Q. E. D.



Sch. This proposition follows evidently from Def. 1, VI., of similar rectilineal figures; it is equivalent to an Axiom, and in this respect agrees with

Prop. 30, I. St. Lines parallel to the same st. line are parallel to each other.

„ 11, VI. Ratios that are the same to the same ratio are the same to one another;

variations of the General Principle,—Things equal to the same thing are equal to one another.

Prop. 22.—Theor.

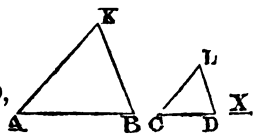
If four st. lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and conversely, *if the similar rectilineal figures similarly described upon four st. lines be proportionals, those st. lines shall also be proportionals.*

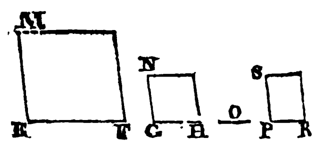
Con. 11, VI. 12, VI. 18, VI.

Dem. 11, V. 22, V. Cor. 2, 20, VI. 9, V. Magnitudes which have the same R to the same M are eq. to one another; and those to which the same M have the same R are eq. to one another.

7, V. Eq. Ms have the same R to the same M; and the same has the same R. to eq. Ms.

Case I. *Let four st. lines be proportionals, &c.*





E. 1	Hyp. 1.	Let AB : CD = EF : GH;
2	,, 2.	on AB & CD let sim. rect. figs. KAB & LCD, be similarly described;
3	,, 3.	& on EF & GH sim. rect. figs. MF & NH, be also similarly described;
4	Conc.	then KAB : LCD = MF : NH.
C.	11, VI.	To AB, CD take a 3rd propl. X, & to EF, GH a 3rd propl. O.
D. 1	H. 1.	∵ AB : CD = EF : GH,
	11, V.	∴ CD : X = GH : O;
2	22, V.	∴ *ex æq.* AB : X = EF : O.
3	Cor. 2,	but AB : X = fig, KAB : fig. LCD;
	20, VI.	and EF : O = fig. MF : fig. NH;
4	11, V.	∴ fig. KAB : fig LCD = fig. MF : fig. NH.

CASE II. *If the sim. rectilineal. figures, &c.*

E. 1	Hyp.	Let fig. KAB : fig. LCD = fig. MF : fig. NH;
2	Conc.	then AB : CD = EF : GH.
C. 1	12, VI.	Make AB : CD = EF : PR,
2	18, VI.	and on PR descr. fig. SR sim. and similarly situated to MF or NH.
D. 1	C. 1.	∵ AB : CD = EF : PR;
2	C. 2.	and on AB, CD are sim. and sim. desc. figs. KAB, LCD;
3	C. 2.	and on EF, PR „ „ figs. MF and SR.
4		∴ figs. KAB : LCD = MF : SR.
5	Hyp.	But KAB : LCD = MF : NH;
6	9, V.	∴ rect. fig. NH = rectl. fig, SR.
7	C. 2.	and ∵ they are sim. and sim. situated, ∴ GH = PR.
8	C. 1. D. 7.	And ∵ AB : CD = EF : PR, and PR = GH;
9	7, V.	∴ AB : CD = EF : GH.
10	Rec.	*If therefore four st. lines be proportionals, &c.* Q .E. D.

COR. As a particular case, *if four st. lines* A,B,C,D, *be proportionals their squares* A^2, B^2, C^2, D^2, *shall also be proportionals*; and conversely; *i. e.*,

$$\text{if } A : B = C : D; \text{ then } A^2 : B^2 = C^2 : D^2;$$
$$\text{and if } A^2 : B^2 = C^2 : D^2; \text{ then } A : B = C : D.$$

SCH. Thus, this proposition is equivalent to the theorem, that, "if two ratios be equal, their duplicates and subduplicates will also be equal," or in other words their powers and roots.

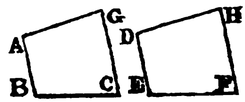
USE & APP. The principle contained in this Theorem is often employed in Arithmetic and Algebra; for if a 3 : b 4 = c 6 : d 8 ; then a^2 9 : b^2 16 = c^2 36 : d^2 64. *If four quantities or numbers are in proportion, their like powers or roots are also proportionals.*

Lemma.—Theor.

If rectilineal figures be equal and similar their homologous sides are equal.

Dem. 16, V. 20, VI.

E.	1	Hyp. 1.	Let fig. AC be =& sim. to fig. DF;
	2	,, 2.	& let BC : BA = EF : ED;
	3	Conc.	then side EF = side BC.
D.	1	Sup.	If EF $\neq$ BC, let EF $>$ BC.
	2	H. 2.16, V.	$\because$ EF : ED = BC : BA; $\therefore$ *altern.* EF : BC = ED : BA;
	3	D. 1. 20, VI	but EF $>$ BC $\therefore$ ED $>$ BA; $\therefore$ fig. DF $>$ fig. AC;
	4	H. 1.	now fig, DF also = fig. AC; —which is impossible;
	5	Conc.	$\therefore$ EF not $\neq$ BC; *i. e.*, EF = BC. Q. E. D.



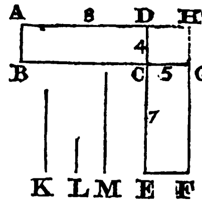
Prop. 23.—Theor.

Equiangular parallelograms have to one another the ratio which is compounded of the ratio of their sides.

Con. 14, I. 31, I. 12, VI.

Dem. Def. A. V. of Compound Ratio. When there are any no. of Ms of the same kind, the 1st is said to have to the last of them the R compounded of the R. which the 1st has to the 2nd, and of the R which the 2nd has to the 3rd, and of the R. which the 3rd has to the 4th, and so on unto the last magnitude. 1, VI. 11, V. 22, V. *ex æquali.*

E. 1	Hyp.	Let ▱ AC be eq. ang. to ▱ CF, having $\angle$BCD = $\angle$ECG;
2	Conc.	then ▱ AC : ▱ CF = R compounded of the ratio of the sides, BC : CG & DC : CE.
C. 1	*Pon.* 14, I.	Place BC & CG in a st. line and also DC & CE;
2	31, I.	complete the ▱ DG; and take a st. line K;
3	12, VI.	make BC : CG = K : L; & DC : CE = L : M.
D. 1	C. 1.	Now K : L & L : M = BC : CG & DC ; CE ;
2	Def. A. V.	but K : M is compounded of K : L & L : M;
3		∴ K : M is a R compounded of the R of the sides.
4	1, VI, C. 3.	And ∵ BC : CG = ▱ AC : ▱ CH, but BC : GC = K : L;
5	11, V.	∴ K : L = ▱ AC : ▱ CH.
6	C. 3	Again ∵ DC : CE = ▱ CH : ▱ CF, but DC : CE = L : M;
7	11, V.	∴ L : M = ▱ CH : ▱ CF.
8	D. 5 & 7,	And ∵ K : L = ▱ AC : ▱ CH;
9	22, V.	∴ *ex. æq.* K : M = ▱ AC : ▱ CF.
10	Remk.	But K : M is compounded of the R of the sides;
11		∴ ▱ AC : ▱ CF is compounded of the Rs of the sides; *i. e.* of BC : CG and DC : CE.
12	Rec.	∴ *Equiangular parallelograms have &c..* Q. E. D.



N, B. This 23rd proposition would follow as a Corollary from the Theorem, " that any two rectangles are to one another in the ratio which is compounded of the ratios of the sides."

A more brief demonstration would be,

E.	Hyp.	Let AC & CF be two eq. ang. parallelograms;
C.	31, I.	Complete the ▱ CH.

D. 1	Def. A, V.	∵ ▱ AC : ▱ CF has the R compounded of the ratio of AC : CH, and of CH : CF;
2		but ▱ AC : ▱ CH = BC : CG, and ▱ CH : ▱ CF = DC : CE;
3	Conc.	∴ ▱ AC : ▱ CF is a ratio compounded of Rs which are the same with the Rs of the sides.

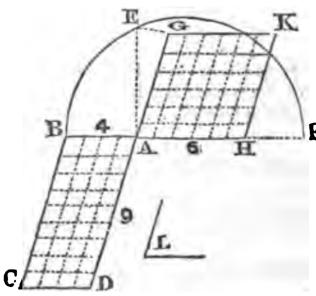
Arith. Hyp. Let BC = 8, CG = 5, DC = 4; and CE = 7.
then 4 : 7 = 5 : 8.75; and of the three numbers 8, 5, & 8.75,
8 : 5 = ▱ 4 × 8 : ▱ 4 × 5;
and 5 : 8.75 = ▱ 4 × 5 : ▱ 4 × 8.75;
thus 8 : 8.75 is the ratio compounded of both; or 32 : 35.

Cor. 1. *If the terms of two analogies are lines, the rectangles under their corresponding terms are proportional.*

Cor. 2. *Hence, also, Rectangles whose bases are proportional and also their altitudes, are themselves proportionals.*

Use & App. *To describe a rhombus equal to a given rectilineal figure* R. *and having an angle equal to a given angle* L.

C. 1	45, I. 23, I.	Draw ▱ AC = given figure R. and with ∠BAD = given ∠L;
2	Pst. 2, I.	Produce BA, DA, indefinitely to G & F;
3	3, I. Pst. 3, I.	Make AF = AD; and on BF desc. a semicircle;
4	11, I.	from A raise a perp. AE cutting the semicircle in E.
5	3, I.	make AG, AH each = AE;
6	Sol.	and AG, AH are sides of the required rhombus.



D. 1	C. 1, 2. &c.	∵ the ▱s AC, AK are equiangular;
2	23, VI.	∴ ▱ AC : ▱ AK has a ratio compounded of the ratio of the sides;
3	Remk.	Thus AB : AH or AE = AG or AE : AD;
4	17, VI.	∴ AG^2 = AB . AD.
5	Remk.	Now AG is a side of the rhombus AK;
6	Conc.	∴ the rhombus AK = the rhomboid AC. Q. E. F.

Prop. 24.—Theor.

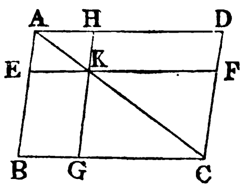
Parallelograms about the diameter of any parallelogram, are similar to the whole and to one another.

Dem. 29, I. If a st. line fall upon two ‖ st. lines, it makes the alternate ∠s = to one another; and the ext. ∠ = the int. and opp. ∠ upon the same side; and likewise the two int. ∠s upon the same side = 2 rt. ∠s. 4. VI.

34, I. The opp. sides and ∠s of ▱s are eq. to one another, and the diam. bisects them. 7. V. Def. 1, VI.

21, VI. Rectl. figures sim. to the same, are similar to each other.

E. 1	Hyp. 1.	Let BD be a ▱ of which AC is the diam;
2	„ 2.	& EH, GF ▱s about the diam.;
3	Conc.	then ▱ EH is sim. to ▱GF; & each sim. to ▱BD.



D. 1	H. 29, I.	∵ BC ‖ EF, ∴ ∠ABC = ∠AEF;
2	H. 29, 1.	& ∵ DC ‖ HK, or HG, ∴ ∠ADC = ∠AHK;
3	34, I.	And ∵ ∠DCB = ∠BAD, & ∠HKE = ∠BAD;
4		∴ ∠DCB = HKE = ∠BAD;

D.	5	Conc.	∴ ▱s BD & EH are equiangular.
	6	D. 2.	And ∵ ∠ADC = ∠AHK, & ∠DAC com.;
	7	4, VI.	∴ △DAC is eq. ang. to △HAK; & ∴ AD : DC = AH : HK.
	8	34, I.	And ∵ opp. sides of ▱s are equal;
	9	7, V.	∴ AD : AB = AH : AE; BC : CD = EK : KH; & BC : BA = KE : EA;
	10	Def. 1, VI.	∴ sides of ▱s BD, & EH about eq ∠s are proportionals.; & ▱ BD sim. to ▱ EH.
	11	*Sim.*	So ▱ BD is sim to ▱ GF.
	12		∵ ▱s EH & GF are each sim to ▱BD;
	13	21, VI.	∴ ▱ EH is sim to ▱ GF.
	14	Rec.	Wherefore *parallelograms about &c.* Q. E. D.

Sch. Proposition 24 should have changed places with Prop. 25.

Use & App. In Perspective this proposition is available to show that a copy is drawn like the original, by the help of a parallelogram.

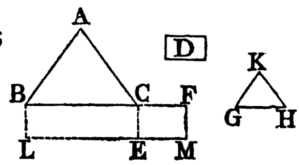
Prop. 25.—Prob. (*Of extensive use.*)

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure, i. e. in area.

Con. Cor. 45, I. To desc. a ▱ equal to a given rectil. fig. and having an ∠ equal to a given rectil. ∠.
29, I. 14, I. 13, VI. 18, VI.

Dem. Cor. 2, 20, VI. 1, VI. 11, V. 14, V. If the 1st has the same R to the 2nd which the 3rd has to the 4th; then, if the 1st be > = or < the 3rd, the 2nd shall be > = or < the 4th.

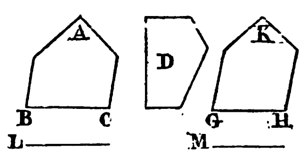
E.	1	Data.	Given rectil. fig. ABC, and rectil. fig. D;
	2	Quæs.	to desc. a rectl. fig. sim. to ABC, and eq. to D.



C. 1	Cor. 45, I.	On BC descr. ▱ BE = fig. ABC;
2	”	and on CE, ▱ CM = fig. D, and having ∠ FCE = ∠ CBL.
3	29, I. 14, I.	Now BC, CF are in a st. line BF, and LE, EM, in st. line LM;
4	13, VI.	find then GH a *mean* propl. to BC, and CF;
5	18, VI.	and on GH descr. a rectl. fig. KGH, sim. to ABC, and similarly situated,
6	Sol.	then the fig. KGH is the rectl· fig. required.
D. 1	C. 4.	∵ BC : GH = GH : CF;
2	Cor.2.20,VI	∴ BC : CF = fig. ABC : fig KGH;
3	1, VI.	but BC : CF = ▱ BE : ▱ EF;
4	11, V.	∴ fig. ABC : fig. KGH, = ▱ BE : ▱EF.
5	C. 1. 14, V.	And ∵ fig. ABC = ▱ BE, ∴ fig. KGH = ▱ EF;
6	C.	but fig. EF = fig. D, ∴ KGH = D and is sim. to ABC;
7	Sol.	Therefore fig. KGH is sim. to ABC and equal to D. Q. E. F.

Or, varying the figures.

C. 1	14, II.	Find L,M, the sides of squares eq. to the figures ABC & D;
2	12, VI.	take GH a 4th propl. to L, M, & BC;
3	18, VI.	and on GH desc. the fig. KGH sim. to ABC.



D. 1	20, VI.	∵ fig. ABC : fig. KGH = $BC^2 : GH^2$; *i. e.*, as $L^2 : M^2$;
2	C. 1.	and ∵ fig. ABC = L^2, and fig. KGH = M^2 = fig. D;
3	C. 3.	and ∵ also KGH is sim. to fig. ABC.
4	Conc.	∴ KGH is the fig. required.

SCH. The chief point of this problem is to find, as in the last fig. but one, a mean proportional to BC, a side of the given figure to which a similar one is to be constructed, and to CF, a side of the rectangle made equal to fig. D;— that mean proportional is GH, which becomes the side of a figure similar to ABC, and equal to D.

USE & APP. By this proposition, while we keep always the same area, we may change the form or shape of the figure,—a process of great use in Practical Geometry; and especially convenient, if we do, what has very often to be done; *i. e.*, reduce an irregular rectilineal figure to its equivalent square.

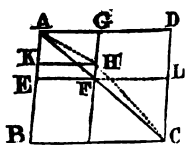
PROP. 26.—THEOR.

If two similar parallelograms have a common angle, and be similarly situated; they are about the same diameter;

"If from a parallelogram a parallelogram, be taken away similar to the whole, and similarly placed and having a common angle with it,—it is about the same diameter with the whole."—EUCLID.

CON. Pst. 2, I. 31, I. DEM. 24, VI. Def. 1, VI. 11, V. 9, V. Ax. 9, I.

E. 1	Hyp. 1.	Let ▱ BD be sim. to ▱ EG, having ∠ A com.;
2	„ 2.	& let AE, AG, sides of ▱ EG fall on AB, AD, homologous sides of ▱ BD;
3	Conc.	then diags. AF, AC are in one line, *i. e.* ▱s BD & EG are about the same diam.:



C. 1	Sup. 1.	If not, and it be possible, let AHC be the diag. of ▱ BD;
2	„ 2.	& AHC be different from AF the diag. of ▱ EG;
3	Pst. 2, I,	let GF meet AHC in H,
4	31, I.	& through . H draw HK ∥ AD or BC.

D. 1	C. 1—4.	∵ ▱ s BD & KG are about the same diag. AHC;
2	24, VI.	∴ ▱ KG is sim. to ▱ BD;
3	Def.1, VI.	& DA : AB = GA : AK;
4	H. Def. 1, VI.	but ∵ ▱ BD is sim. to ▱ EG, ∴ DA : AB = GA : AE;
5	11, V.	∴ GA : AE = GA : AK;
6	9, V. Ax. 9	& ∴ AK = AE,—which is impossible:
7	Conc.	∴ diag. of ▱ BD through ∠ A cannot be otherwise than on AF, *i. e.*, AF & AC are in the same st· line.
8	Rec.	Therefore, *If two similar parallelograms &c.* Q. E. D

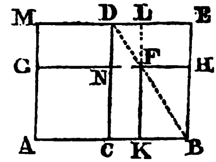
Sch.—This Proposition is the *converse* of Prop. 24, and properly should follow it immediately, or be incorporated with it.

Prop. 27.—Theor.

Of all parallelograms applied to the same st. line, and deficient by parallelograms similar and similarly situated to that which is described upon the half of the line; that which is applied to the half and is similar to the defect, is the greatest.

N. B. More easily to understand this and the two following propositions, attention must be given to the Subsidiary Definitions D, E, & F. p. 251, 2.

Con. Pst. 1. I. 31, I. Dem. 43, I. The complements of the ▱s which are about the diam. of any ▱, are eq. to one another. 26. VI. Axs. 1, 2, 9. I. 34, I. 36, I. ▱s upon eq. bases, and between the same ‖s are eq. Ax. 4, I.

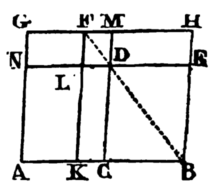


E. 1	Hyp. 1.	Let the st. line AB be bisd. in C;
2	„ 2.	& on the half AC let a ▱ AD be applied, deficient from ▱ AE on the whole line AB, by the ▱ CE on the other half line CB;
3	Conc. 1.	then, of all ▱s applied to any other pts. of AB, deficient by ▱s sim. &similarly situated to ▱ CE, ▱ AD shall be the greatest.
4	Hyp. 3.	Let any ▱ AF be applied to AK, a pt. of AB and $\neq$ AC or CB, so as to be deficient from ▱ AH on AB by ▱ KH,— ▱ KH being sim. and similarly situated to ▱ CE;
5	Conc 2.	then ▱ AD on AC is $>$ ▱ AF.

CASE I. *Suppose* AK, *base of* ▱ AF, $>$ AC *the half of* AB; and $\because$ by Hyp. 3, ▱ CE is sim, to ▱ HK,
$\therefore$ by 26, VI., they are about the same diam. DB.

C.	31, I.	Draw diag. DB, and complete the scheme.
D. 1	43, I.	$\because$ ▱ CF = ▱ FE, to each add ▱ KH,
2	Ax 2, I.	$\therefore$ the whole ▱ CH = the whole ▱ KE;
3	H. 1. Ax. 1.	But $\because$ AC = CB, $\therefore$ ▱ CH = ▱ CG = KE;
4	*Add.* Ax. 2,	to each add CF, $\because$ ▱ AF = gnomon CHL;
5	Ax. 9.	$\therefore$ ▱ CE or AD $>$ ▱ AF.

CASE II. *Let AK, base of* ▱ *AF, be* $<$ *AC, the half of AB.*



C.	31, I.	A similar construction is to be made.
D. 1	H. 1. 34, I.	$\because$ BC = CA, $\therefore$ HM = MG;
2	36, I. Ax. 9.	and $\because$ ▱ DH = ▱ DG. $\therefore$ ▱ DH $>$ ▱ LG.
3	43, I.	but ▱ DH = ▱ DK, and ▱ DK $>$ ▱ LG;
4	*Add.* Ax. 4.	to each ad ▱ AL, $\therefore$ ▱ AD $>$ ▱ AF.

D. 5	Conc.	Thus in both cases ▱ AD > ▱ AF.
6	Rec.	Therefore, *of all ▱s applied to* AB, *and deficient by ▱s each sim. and similarly situated to* AB, *the* ▱ AD *is the greatest.* Q. E. D.

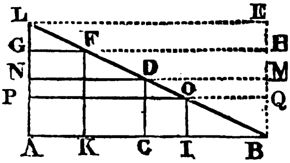
Otherwise the Proposition may be enunciated;—*Of all the rectangles contained by the segments of a given st. line, the greatest is the square which is described on half the line.*

E. 1	Hyp.	Given AB bisected in C and unequally divided in D;
2	Conc.	then AC^2 > ▭ AD . DB.
D. 1	H.	∵ AB is div. equally in C and uneq. in D;
2	5, II.	∴ $AD . DB + CD^2 = AC^2$
3	Conc.	∴ $AC^2 > AD . DB$.

SCH. DE CHALES, LARDNER, and some other geometricians recommend that Propositions 27, 28, and 29 should be omitted as unnecessary; but they were frequently employed by the ancient mathematicians, and are required especially for the solution of several problems.

USE AND APP.—*In a given* △ABL *to inscribe the greatest parallelogram possible, having an angle, A, in common with the triangle.*

C. 1	31, I.	Complete the ▱ AE of which LB is the diag.
2	10, I.	bis. AB in C, AC in K; AL in N and LN in G;
3	31, I.	and through . s K, C, I, draw ‖s to AL,
4	,,	and through . s G, N, P draw ‖s to AB.
5	Sol.	then parallelog. AD, with ∠A com. to △ABL, is the *greatest* parallelog. possible in △ABL.



D. 1	24, VI.	∵ the defects of parallelogs AO, AD &c. are parallelogs. sim. to parallelog. CM ;
2	26, VI.	∴ those parallelogs. are about the same diag. BL ;
3		∴ pargs. AO, AD, AF &c. are inscribed in the given △ABL;
4	C. 2-4.	And ∵ parallelog. AD is described on AC half the base,
5	27, VI.	∴ parallelog. AD is the greatest parallelog. in the △ABL, and the ∠A is com. to the parallelog. & the △. Q. E. F.

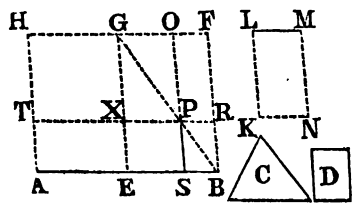
Prop. 28.—Prob.

To a given st. line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram; but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.

"To given rt. line to apply a parallelogram equal to a given rectilineal figure and deficient by a figure similar to the given parallelogram."—Euclid.

Con. 10, I. To bis. a given finite st. line, 18, VI. 31, I. 25, VI. 3, I.

Dem. 21, VI. 26, VI. Ax. 3. 43, I. 36, I. Ax. 1. 24, VI.



E. 1	Dat. 1.	Let AB be the given st. line, and C the given fig. to which the parallelogram is to be equal;
2	„ 2.	let fig. C be $\ngtr$ the ▱ on the half line AE, of which the defect is sim. to the defect of that ▱ which is to be applied;
3	„ 3.	and let fig. D be the ▱ to which this defect is to be similar.
4	Quæs.	To apply to AB a ▱ = C, and deficient from the ▱ on the line AB, by a ▱ sim. to fig. D.
C. 1	10, I. 18, VI.	Bis. AB in E; and on EB desc. ▱ EF sim. and similarly situated to fig. D;
2	31, I.	complete the ▱ AG, either = C, or > C.

Case I. *Suppose ▱ AG = fig. C*;

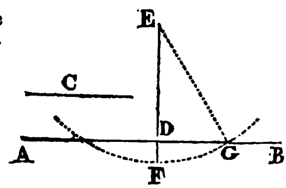
D.	Conc.	then on AB is applied ▱ AG = fig. C, and deficient by ▱ EF sim to fig. D. Q.E.D.

Case II. *Suppose ▱ AG $\neq$ fig. C, but > C*; and by 36, I, ▱ EF = ▱ AG, & ∴ ▱ EF also > fig. C.

C.	25, VI.	Make ▱ KM = excess of ▱ EF above fig. C, & sim. and simly situated to fig. D.
D. 1	C.1.21, VI	∵ fig D is sim. to ▱ EF, ∴ ▱ KM is sim. to ▱ EF.
2	Sup.	Let KL be homol. to EG, & LM to GF.
3	C. 3.	And ∵ ▱ EF = C + KM, ∴ EF > KM,
4		∴ line EG > KL, & GF > LM.
5	3, I. 31, I.	Make GX = LK, GO = LM; and complete ▱ XO.
6	C. D. 1.	∵ ▱ XO is = & sim. to ▱ KM, & KM sim. to ▱ EF,
7		∴ ▱ XO is sim. to ▱ EF,
8	26, VI.	& ∴ ▱s XO & EF *both* are about diag. GB.
9	Sup. 31, I.	Let GPB be their diag.; and complete the scheme.
10	D. 3, 6.	then ∵ ▱ EF = C + KM, and a part ▱ XO = ▱ KM a part of the other;

D.11	Ax. 3.	∴ rem. gnomon ERO = rem. C.
12	43, I.	And ∴ ▱ OR = parlm. XS, ∴ on adding ▱ SR to each, ▱ OB = parlm. XB;
13	C. 1, 36, I. Ax. 1.	but AE = EB; ∴ parlm. XB = parlm. TE & parlm. TE = parlm. OB;
14	*Add.*	to each add parlm. XS; ∴ the whole ▭ TS = the gnomon ERO;
15	D. 11.	but ERO = fig. C; ∴ also parlm. TS = fig. C.
16	Rec. 24, VI.	∵ *parlm.* TS = fig. *C is applied to st. line* AB, *deficient by parlm. SR sim. to fig. D, because SR is sim. to EF.* Q. E. F.

Sch. 1. The Proposition may be thus enunciated; *To divide a given st line AB, so that the rectangle contained by the segments may be equal to a given space, as the square on C; but that given space must not be greater than the square of half the given line.*



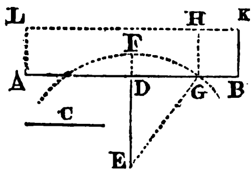
E. 1	Dat.	Given st. line AB, & C^2 the space to which the seg. of AB must be equal, but $C^2 \ngtr \left(\frac{AB}{2}\right)^2$
2	Quæs.	to divide AB so that the ▭ contained by the segs = C^2.
C.	10, I.	Bis. AB in D.
D. 1	Sup. 1.	If $AD^2 = C^2$ the Problem is solved;
2	Sup. 2.	but if $AD^2 \neq C^2$, & AD > C;
3	Sim.	then the Solution though differing in form will be like the foregoing Prop. 28 in substance.
C. 1	3, I. 11, I.	At rt. ∠s to AB draw DE = C;
2	3, I.	& prod. ED so that EF = AD or DB;
3	Pst. 3 & 1, I	from E with rad, EF cut AB in G, and join EG;
4	Sol.	then AB is div. in G so that AG . GB = C^2
D. 1	C.	∵ AB is divided equally in D and unequally in G;
2	5, II.	∴ ▭ AG . GB + $DG^2 = DB^2 = EF^2 = EG^2$;
3	47, I.	but $ED^2 + DG^2 = EG^2$;
4	Ax. 1. 1.	∴ ▭ AG . GB + $DG^2 = ED^2 + DG^2$;
5	*Sub.* Ax. 3	take away DG^2, & ▭ AG . GB = $ED^2 = C^2$. Q. E. F.

Sch. 2. Prop. 28. bk. VI. is equivalent to the Problem;—"*To inscribe in a given △ a parlm. equal to a given figure not greater than the maximum inscribed parlm. and having an ∠ in common with the △.*" *Manual of Euclid,* Pt. II. *p.* 98.

The Demonstration is similar to that in Use and App. 27, VI.

Use & Appl. There are two cases of this Proposition which are not unfrequently employed by geometers.

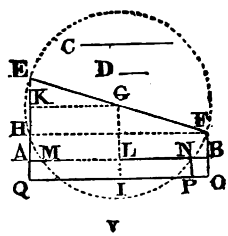
1°. A variation of Sch. 1, 28, VI. *To a given st. line* AB *to apply a rect.* AH, *deficient by a square* GK, *which rect. shall be equal to a given square, that on line* C; *but the given square on* C *must not be greater than the square on the half* AD *of line* AB.



C. 1	10, I.	Bis. AB in D; and if $AD^2 = C^2$, the prob. is solved.
2	Sup.	But if $AD^2 \neq C^2$ and $AD > C$;
3	11, I. 3, I.	draw DE ⊥ AB, and make DE = C;
4	3, I.	produce ED so that EF = AD or DB;
5	Pst. 3 & 1, I	from cen. E, with EF desc. arc meeting AB in G; and join EG;
6	46, I. 31, I,	on GB desc. the square GK, and complete rect. GL;
7	Sol.	then ▭ AH = C^2, and deficient by GK, has been applied to AB.
D. 1	C.	∵ AB is div. equally in D, and unequally in G;
2	5, II. 47, I,	∴ $AG.GB + DG^2 = DB^2 = EF^2 = EG^2 = ED^2 + DG^2$.
3	Sub. Ax. 3.	Take DG^2 from each, ∴ $AG.GB = ED^2$, *i. e.* $= C^2$.
4	C. 6.	But AG . GB is ▭ AH, ∴ GH = GB;
5		∴ ▭ AH = C^2.
6	Conc.	∵ ▭ AH = C^2 has been applied to AB, deficient by the sq. GK. Q. E. F.

2°. *To a given st. line to apply a rectangle, which shall be equal to a given rectangle, and be deficient by a square; but the given rectangle must not be greater than the square upon half the given line.*

E.	Dat.	Let AB be the given line, and on lines C & D a given ▭ not greater than $\left(\frac{AB}{2}\right)^2$;
2	Quæs.	to apply to AB a ▭ = C . D, deficient by a square.
C. 1	11, I.	On the same side of AB, at A & B, draw AE, BF each ⊥ AB;
2	2, I. 10, I.	make AE = C and BF = D; join EF and bisect it in G;
3	Pst. 3, I.	from G, with rad. GE, desc. a ⊙ meeting AE in H;



4	31, I.	join HF; and draw GK ∥ HF, and GL ∥ AE, meeting AB in L.
D. 1	31, III. Ax. 11, I.	∵ ∠ EHF in a sem. c. is a rt. ∠, and = ∠ EAB;
2	28, I. 34, I.	∴ AB ∥ HF and AH ∥ BF, ∴ AH = BF;
3	Rem. C. 2.	and ▭ EA . AH = ▭ EA . BF, *i. e.*, C.D.
4	C. 2, 4.	and ∵ EG = GF, and AE ∥ LG ∥ BF;
5	34, I. 3, III.	∴ AL = LB, & EK = KH; and C . D ≯ AL^2 or $\left(\frac{AB}{2}\right)^2$;
6	Conc.	∴ EA . AH ≯ AL^2, *i. e.* KG^2;
7	*Add.* 47, I.	add KE^2, ∴ AK^2 ≯ $EK^2 + KG^2$. *i. e.* EG^2;
8		and ∴ AK or GL ≯ GE.
9	Def. 2, III.	Now if GE = GL, ⊙ EHF touches AB in L;
10	36, III.	∴ AL^2 = EA . AH, *i. e.* C . D.
11	Sup.	But if EG ≠ GL, and EG > GL; ∴ ⊙ EHF cuts AB;
12	Sup.	Let the ⊙ cut AB in . s M and N;
13	46, I. 31, I.	on NB desc. NB^2 and complete the ▭ AP.
14	3, III. Ax. 3	∵ LM = LN, and AL = LB, ∴ AM = NB,
15	C.	and ∴ AN . NB = NA . AM, *i. e.* EA . AH or C . D.
16	C.	But AN . NB is ▭ AP; ∴ PN = NB;
17	Rec.	∴ ▭ AP = C . D, and AP has been applied to AB, deficient by BN^2. Q. E. F.

This last Problem may be thus enunciated; "*To cut a given line* AB *in the point* N *so as to make the rectangle* AN . NB *equal to a given space.*"

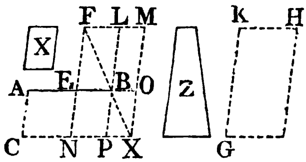
Or, which is the same thing, "*Having* AB *the sum of the sides of a rectangle given, and also its magnitude, or area, to find the sides.*"

Prop. 29.—Prob.

To a given st line to apply a parallelogram equal to a given rectilineal figure, and exceeding by a parallelogram similar to another given parallelogram.

Con. 10, I. 18, VI. 25, VI. 21, VI. 2, I. 31, I.
Dem. 26, VI. 36, I. 43, I. 24; VI.

E. 1	Dat.	Given AB, ▱ X, & rectil. fig. Z,
2	Quæs.	to apply to AB a ▱ = Z, and exceeding by a ▱ sim. to ▱ X.



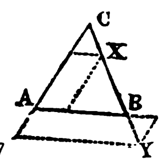
C. 1	10, I. 18, VI.	Bis. AB in E, and on EL descr. ▱ EL sim. & sim. sit. to ▱X;
2	25, VI.	make ▱ GH = EL + Z, & sim. & sim. to ▱ X.
3	21, VI.	∴ ▱ GH is sim. to ▱ EL.
4		Let KH be homol. to FL, & KG to FE;
5	C. 2.	then ∵ ▱ GH > ▱ EL, ∴ HK > FL & KG > FE;
6	Pst. 2, I. 3, 1.	Produce FL & FE, & make FLM = KH, & FEN = KG;
7	31, I.	& complete the ▱ MN;
8	C, 6, 7.	∴ ▱ MN is eq. & sim. to ▱ GH;
9	C. 3.	but GH sim to EL, ∴ MN is sim to EL;
10	26, VI.	& ∴ EL & MN are ▱s about the same diag.
11	31, I.	Draw their diag. FX, & complete the scheme,
12	Sol.	then to AB is applied ▱ AX = Z, & exceeding by ▱ PO sim. to ▱ X.
D. 1	C. 2, 8. Ax. 1.	∵ GH = EL + Z, & ▱GH = ▱MN, ∴ ▱ MN = ▱EL + ▱ Z;
2	*Sub.* Ax. 3.	take away ▱ EL; ∴ rem. gnomon NOL = Z.
3	C. 1, 31, I. 43, I.	And ∵ AE = EB ∴ ▱ AN = ▱ NB ▱ BM.
4	*Add.* Ax. 2.	Add ▱ NO, ∴ the whole ▱ AX = the whole gnomon. NOL;
5	D. 2.	but NOL = Z, ∴ ▱ AX = fig. Z.
6	Conc.	∴ to AB is applied a ▱ AX = fig. Z, exceeding by ▱ PO, sim. to fig. X; ∵ ▱ PO is sim. to ▱ EL. Q. E. F.

N.B.—In the diagram, X by mistake of the engraver occurs twice.

SCH.—Of the thirteen books of Elements written by Euclid, the tenth bears evident traces of the greatest attention having been bestowed to render it complete. The doctrine of *Incommensurables* there receives its developement, and is treated with great exactness. "The most conspicuous propositions of elementary geometry," says an eminent writer, "which are applied in the tenth book, are the 27th, 28th, & 29th of the sixth book, of which it may be useful to give the algebraical signification. The first of these (the 27th) amounts to shewing that $2x - x^2$ has its greatest value when $x = 1$, and contains a limitation necessary to the conditions of the two which follow. The 28th proposition is a solution of the equation $ax - x^2 = b$, upon a condition derived from the preceding proposition, namely, that $\frac{1}{4}a^2$ shall exceed b. It might appear more correct to say that the solution of this equation is one particular case of the proposition, namely, where the given parallelogram is a square ; but nevertheless, the assertion applies equally to all cases. Euclid, however, did not detect the *two* solutions of the question ; though if the diagonal of a parallelogram in his construction be produced to meet the production of a line which it does not cut, the second solution may be readily obtained. This is a strong presumption against his having anything like algebra ; since it is almost impossible to imagine that the propositions of the tenth book, deduced from any algebra, however imperfect, could have been put together without the discovery of the second root. The remaining proposition (the 29th) is equivalent to a solution of $ax + x^2 = b$; but the case of $x^2 - ax = b$ is wanting, which is another argument against Euclid having known any algebraical reasoning.— PENNY CYC. XII, p. 38.

USE and APP.—Several Problems of a like kind to Prop. 29, and in some respects equivalent to it may be here advantageously introduced ;

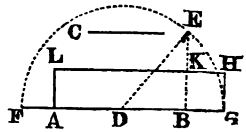
PROB. 1°. *exscribeTo to a given triangle ABC, a parallelogram equal to a given rectilineal figure, AX, and having an angle equal to one of the angles of the given triangle.*



Here AX is the *inscribed* ▱ & AY the *exscribed* ▱

N.B. This Prob. is equivalent to to the foregoing Prop. 29. *Manual of Euclid*, p. 100.

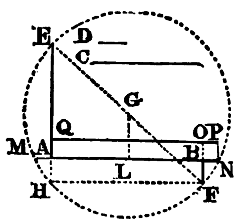
PROB. 2°. To *a given st. line AB to apply a rectangle which shall be equal to a given square, that on C and exceeding by a square..*



C. 1	10, I. 11, I.	Bis. AD in D, and draw BE $\perp$ to AB so that BE $=$ C;
2	Pst. 1, 3, I.	join ED, and from D with DE desc. a $\odot$ meeting AB produced in G;
3	36, I, 31, I.	on BG desc. the sq. BH, and complete $\square$ GL
4	Sol.	then $\square$ AH $=$ C^2, & exceeding by GK is applied to AB.
D. 1	C.	$\because$ AB is divided equally in D and produced to G,
2	6, II	$\therefore$ AG . GB $+$ DB^2 $=$ DG^2 $=$ DE.^{2} $=$ EB^2 $+$ BD^2;
3	*Sub.* Ax. 3, I.	from each take DB^2, $\therefore$ rem. AG . GB $=$ BE^2 $=$ C^2
4	C. 3.	but $\because$ GH $=$ GB, $\therefore$ AG, GB is rectangle AH;
5		$\therefore$ $\square$ AH $=$ C^2
6	Conc.	$\therefore$ AH $=$ C^2, and exceeding by GK, is applied to AB.

Q. E. F.

N.B.—This Problem is the same as, "*To produce a given st. line, AB, so that the rectangle contained by the external segments of the given line may be equal to a given space, as C^2.*" The foregoing Con. and Dem. may be used, and then it will appear that the problem is, *to produce AB so that AG . GB = C^2.*



C. 1	2, I. 11, I.	Draw AE $=$ C, & BF $=$ D, $\perp$s AB on contrary sides;
2	Pst. 1, I. 31, I.	join EF & bisect EF in G;
3	Pst. 3, I.	from G with GE desc. $\odot$ meeting AE in H;
4	Pst. 1, 31, I.	join HF, & draw GL $\parallel$ AE.
5		Let the $\odot$ meet AB produced in M, N;
6	46, I, 31, I.	on NB desc. sq. NO, & complete $\square$ NQ.
7	Sol.	to AB is applied a $\square$ AP $=$ C . D & exceeding by square BP.
D. 1	C. 1, 28, I.	$\because$ $\angle$EHF a rt. $\angle$ $=$ $\angle$ EAB, $\therefore$ AB $\parallel$ HF;
2	34, I.	$\therefore$ AH $=$ BF; & EA . AH $=$ EA . BF $=$ C . D,
3	3, III. Ax. 3, I.	And $\because$ ML $=$ LN, & AL $=$ LB, $\therefore$ MA $=$ BN;
4	35, III.	& $\therefore$ AN . NB $=$ MA . AN $=$ EA . AH $=$ C . D;
5	Ax. 1, I.	$\therefore$ AN . NB, *i.e.*, AP $=$ C . D.
6	Conc.	$\therefore$ to AB is applied AP $=$ C . D & exceeding by sq. BP.

Q. E. F.

N.B. This Problem is the same as,—"*To find a point N in a given st. line AB produced, so as to make the rectangle AN . NB = a given space.*"

Or, which is the same thing,—"*Having given AB the difference of the sides of a rectangle and the magnitude of it, to find the sides.*"

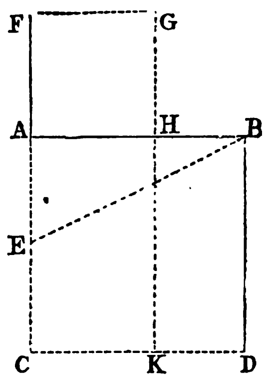
Prop. 30.—Prob.

To cut a given st. line in extreme and mean ratio; i. e., so that the whole line shall be to the greater segment as the greater segment to the less.

Con. 46, I. To describe a sq. on a given st. line.
11, II. To divide a given st. line into two pts, so that the rect. contained by the whole and one of the pts, shall be equal to the sq. of the other part.

Dem. Ax. 3, I. 14, VI. 34, I. Def. 30, I. A sq. has all its sides eq. and all its ∠s rt. ∠s. 14. VI. 17, VI. Def. 3, VI.

E,	1	Dat.	Given AB a st. line;
	2	Quæs.	to cut it in extreme and mean ratio.



C.	1	45, I.	On AB construct a square AD;
	2	29, VI.	to AC apply ▭ CG = ▭ AD, exceeding by fig. AG, sim. to AD;
	3	Sol.	then AB is cut in extreme & mean ratio in . H.
D.	1	C. 1.	∵ AD is a square, ∴ AG is a sq.
	2	C. 2. Ax. 3, I.	∵ sq. AD = ▭ CG, ∴ rem. BK = rem. AG;
	3	C. 14, VI.	& ∵ BK is eq, ang. to AG, ∴ KH: HG = AH : HB;

4	34, I.	but KH = AC = AB;
	Def. 30, I.	& HG = AH;
5		∴ BA : AH = AH : HB;
6	Ax. 9, I.	but AB > AH;
	14, V.	AH > HB.
7	Con. Def.	∴ AB is cut in extreme and mean ratio,
	3, VI.	*i. e.*, AB : AH = AH : HB. Q. E. F

Otherwise,

C.	11, II.	Divide AB in H so that AB . BH = AH^2.
D. 1	C. 17, VI.	∵ AB . BH = AH^2, ∴ AB : AH = AH : HB;
2	Def. 3, VI.	∴ AB is cut in extreme and mean ratio in . H.

SCH. 1. A st. line thus divided is also said to be divided *medially*; an the ratio of its segments is called the *medial ratio*. The same division take place in Prop. 11, bk. II, when the rectangle of the whole and one of its parts is equal to the square on the other part.

2. The dividing a line into extreme and mean ratio belongs to a class of Problems which relate to incommensurable magnitudes. The following is the General Theorem respecting them.

"*Let there be two Magnitudes of the same kind,* P & Q; *and let* P *be contained in* Q *a certain number of times which is to* P *as* P *is to* Q; *then the Magnitudes* P & Q *shall be incommensurable.*

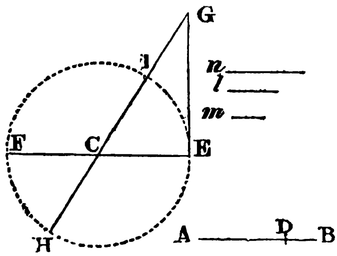
C.	*Sup.*	Let 5 P = Q — R a remainder.
D. 1	Hyp. 15, V.	Then ∵ R : P = P : Q, and 5 R : 5 P = P : Q;
2	16, V. 17, V.	∴ *alt.* 5 R : P = 5 P : Q, and *div.* 5 R ∾ P : P = 5P ∾ Q : Q.
3	16, V. 23, V.	But P : R = Q : P, ∴ *ex æq.* 5 R ∾ P : R = 5 P ∾ Q or R : P.
4	A, V.	Now R < P, ∴ 5 R ∾ P < R; *i. e.* 5 R = P with a rem. 5 R ∾ P or R.
5		And R_2 : R = R : P;
6		∴ 5 R^2 = R, with a rem. R_3.
7	*Sim.*	So 5 times every Rem. = preceding Rem. with a new Rem.;
8	Sch. N, V.	∴ No Rem. is contained exactly in the preceding Rem.;
9	Conc.	∴ P & Q have no common measure.

Now the segments AH, HB of a line AB, medially divided are magnitudes like P and Q; for the gr. seg. AH is contained in the whole line AB once, with a rem. HB; and HB : AH = AH : AB; ∴ HB also is contained once in AH with a rem.; and R_2 : HB = HB : AH; ∴ AH and HB are incommensurable.

3. Or, the dividing of a line into extreme and mean ratio may be considered as a particular case of the following more general problem;

To divide AB *so that the rectangle under the whole line, and one part shall bear a given ratio, as* m : n, *to the square of the other part.*

C.	1	10, I. Pst. 3, I.	With EF as diam. let a ⊙ be described;
	2	13, VI.	take l a mean propl. of m & n,
	3	17, III. 12, VI.	and on tang. at E take EG a 4th propl. to l, m and EF;
	4	Pst. 1, I.	through cen. C, draw GH,
	5	'	and cut AB in D so that AD : DB = HI : IG.
	6	Sol.	then AB will be cut as $m : n$.



D	1	C. 3,	∵ $EG^2 : EF^2 = m : n$.
	2	*Note* †	∴ $EG : EF = m : l$, i. e. in the subduplicate ratio of $m : n$.
	3	36, III.	but $EG^2 = HG \cdot GI$; ∴ $HG \cdot GI : HI^2 = m : n$;
	4		∴ HG is cut as required at I.
	5	C. 5.	and AD : DB = HI : IG,
	6	Conc.	∴ AB also is cut as required in the . D. Q. E. F.

N. B. In the solution of this problem it is assumed that if HI : IG = AD : DB, then $HG \cdot GI : HI^2 = AB \cdot BD : AD^2$. The proof is simple, and is given in LARDNER's *Euclid*, p. 220.

" Hence, *if two lines be cut in extreme and mean R, they are cut similarly*; and *if a line be cut in extreme and mean R, any line cut similarly will be also cut in extreme and mean R.*"

† "The ratio $\frac{A}{B}$ is *subduplicate* of $\frac{a}{b}$ when it is equal to the ratio of a to a mean proportional between a and b.

4. *Numerically,* as in Sch. 2., *we may approximate to the ratio of incommensurables* by a process similar to that of Continued Fractions; thus,

Take *two* magnitudes P & Q; and let 2 P + Rem. (remainder) = Q;
also let Rem. : P = P : Q:

Then, by making *each new term* = 2 *last,* + *the last but one,*
P : Q lies between any two consecutive ratios of the series,
1 : 2, 2 : 5, 5 : 12, 12 : 29, 29 : 70, 70 : 169, &c.,

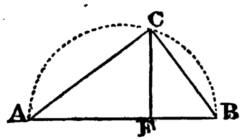
i. e., the rat. P : Q shall lie between 1 : 2, and 2 : 5; again between 2 : 5 and 5 : 12 &c.; the ratios of each successive pair approaching always more nearly to one another, and therefore to the ratio sought.

In the same manner, by making *each new term* = 5 *last* + *the last but one,*
when 5 P + Rem. = Q,
we may approximate to the ratio P : Q by means of the series
1 : 5, 5 : 26, 26 : 135, 135 : 701, &c.

N. B. In the case of *the medial ratio* the series is the simplest possible; namely, 1 : 1, 2 : 3, 3 : 5, 5 : 8, 8 : 13, 13 : 21, 21 : 34, 34 : 45, &c., where each successive term is equal to the sum of the last two.

Use & App. This Proposition is employed in Euclid's 13th Book, treating of the construction of the five regular or Platonic solids,—the Tetrahedron, the Cube, the Octahedron, the Dodecahedron, and the Eicosahedron: it enables us also to solve the following, among other Problems:—

Prob. 1°. *On a given line,* AB, *to construct a rt. ∠d triangle, the sides of which shall be in continued or geometrical progression.*



C.	1	Pst. 3, 1.	On AB construct a semicircle.
	2	30, VI.	divide AB medially in E, *i. e.* AB : AE=AE : EB;
	3	11, I.	from E raise a perp. EC cutting the semic in C;
	4	31, III.	on joining CA & CB, ACB is a rt. ∠;
	5	Sol.	and the sides of △ ACB are continued proportionals, *i. e.* AB : AC = AC : CB.
D.	1	C.2. Def.10, V.	∵ AB : AE = AE : EB, ∴ AE a mean propl.;
	2	8, VI.	and ∵ AB : BC = BC : EB, ∴ CB a mean propl.
	3	9, V.	and ∴ AE = CB.
	4	8, VI	But AB : AC = AC : AE, and AE = CB.
	5	Conc.	∴ AB : AC = AC : CB. *i. e.* the sides of the rt. ∠d △ are in continued proportion. Q. E. F.

N. B. In this Demonstration, steps 1—3, it appears, that if the perp. CE cuts the hypotenuse AB, in extreme and mean ratio, then *the less side* CB = *the alternate segment* AE, and *vice versâ.*

PROB. 2. " *The altitude* EC *of a rt.* $\angle$*d* $\triangle$ *being given, of which the sides are in a given ratio, to find the sides.*

By aid of the alt. EB, and from the ratio of the sides the segments AE and EB may be found; for

E. 1	Dat. 1	Let M : N represent the R. of the greater side to the less.
2	„ 2	& let G & L represent the gr. and less segs· of AB, CE being the altitude ;
3	Quæs.	to find the sides of the rt. $\angle$d $\triangle$.
D. 1	8, VI.	Here M : N = CE : L, & N : M = CE : G.
2	E. 2 & 1.	And $\because$ CE is given, and M : N known ;
3		$\therefore$ G and L the segments may be found ;
4	D. 1.	for $L = \frac{N . AE,}{M}$ and $G = \frac{M . CE.}{N.}$;
5	By last Prob	Now G = the less side, & AB^2 — G = greater side.

Q. E. F.

PROB. 3°, *Given, a, b, two sides of a* $\triangle$, *and the diameter, d, of the circumscribing* circle, to find the other side, *c*.

C. 1		Take $\frac{a\,.\,b}{d}$, then $a^2 - \left(\frac{a\,.\,b}{d}\right)^2$ & $b^2 - \left(\frac{a\,.\,b}{d}\right)^2$;
2		And $\sqrt{a^2 - \left(\frac{a\,.\,b}{a}\right)} + \sqrt{b^2 - \left(\frac{a\,.\,b.}{d}\right)^2}$ *c* the third side
D. 1	U&A 16 VI	For $a\,.\,b = d.\ alt.$; $\therefore$ alt. $= \frac{a\,.\,b\,.}{d}$
2		And a^2 — alt^2 = greater seg.^{2} ; b^2— alt.^{2} = less seg.
3		then $\sqrt{}$ gr. seg.^{2} + $\sqrt{}$ less seg.^{2} = *c*, the third side.

Ex. Two sides of a $\triangle$ measure respectively 8 feet and 6 feet, the diameter of the circumscribing $\odot$ being 10 feet; required the third side.

Here $\frac{8 \times 6}{10} = 4.8$; then $\sqrt{64} - 23.04 = \sqrt{40.96} = 6.4$;

& $\sqrt{36} - 23.04 = \sqrt{12.96} = 3.6$

$\cdot$ $6.4 + 3.6 = 10$ the third side.

PROP. 31.—THEOR. *(Important.)*

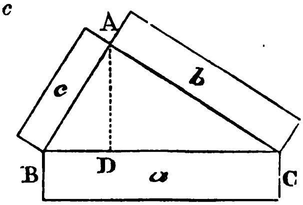
In right angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

CON. 12, I. To draw a st. line ⊥ a given st. line from a given . without it.

DEM. 8, VI. 4, VI. Cor. 2, 20, VI. B, V. If four Ms are propls. they are propls. also when taken inversely.
24, V. If the 1st has to the 2nd the same R. which the 3rd has to the 4th; and the 5th to the 2nd the same R. which the 6th has to the 4th; the 1st and 5th together shall have to the 2nd the same R which the 3rd and 6th together have to the 4th. Ax. 8, I.

A, V. If the 1st of four Ms. has the same R. to the 2nd which the 3rd has to the 4th; then, if the 1st be > = or < the 2nd, the 3rd also is > = or < the 4th.

E. 1	Hyp. 1.	Let ABC be a rt. ∠d △, the rt. ∠ being ∠BAC;
2	„ 2.	and let fig. *a* on BC be sim. and similarly desc. to figures *b* & *c* on AC, AB;
3	Conc.	then fig. *a* = fig. *b* + fig. *c*.
C.	12, 1.	Draw from . A, AD ⊥ BC.
D. 1	C.	∵ from A, the vertex of the rt. ∠ AD ⊥ base BC,
2	8, VI.	∴ △s ABD, ADC, are sim. to △ ABC and to each other.
3	D. 2.	And ∵ △ABC is sim. to △ADB, & to △ADC,
4	4, VI.	∴ CB : BA = BA : BD, & CB : CA = CA : DC;
5	Cor. 2, 20, VI.	and ∵ in 3 propls, as 1st : 3rd so is fig. on 1st : fig. on the 2nd,



6		$\therefore$ CB : BD = fig. *a* on CB : sim. and similarly desc. fig. *c* on BA ;
7	B. V.	and *inv.* DB : BC = fig. *c* on BA : fig. *a* on CB ;
8	*Sim.*	So DC : CB = fig. *b* on CA : fig. *a* on CB ;
9	24, V.	$\therefore$ BD + DC : BC = fig. *c* on BA + fig. *b* on AC : fig. *a* on BC
10	Ax. 8, I.	but BD + DC = BC.
11	A. V.	$\therefore$ fig. *a* on BC = sim. and similarly desc. figs. *c* + *b* on BA and AC.
12	Rec.	Wherefore, *in rt. ∠d triangles, the rectilineal fig. &c.* Q. E. D.

Or,

D. 1	23, VI.	$\therefore$ sim. figs. : one another in the duplicate R. of homol. sides,
2		$\therefore$ fig. *a* on BC : fig. *c* on BA, in the duplicate R of CB to AB ;
3	Cor. 1. 20, VI	but $BC^2 : BA^2 = BC : BA$;
4	11, V.	$\therefore$ fig. *a* on CB : fig. *c* on BA = $CB^2 \cdot AB^2$.
5	*Sim.*	So fig. *a* on BC : fig. *b* on CA = $BC^2 : CA^2$;
6	24, V.	$\therefore$ fig. *a* on BC : figs. *c* + *b* on BA, AC = $BC^2 : AB^2 + AC^2$;
7	47, I.	but $BC^2 = BA^2 + AC^2$;
8	Conc.	$\therefore$ fig. *a* on BC = fig. *c* on BA + fig *b* on AC. Q. E. D.

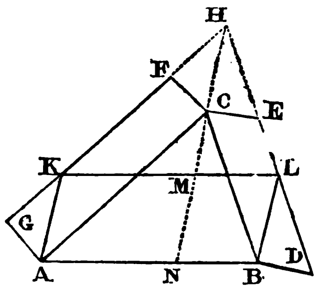
Or,

$$\text{Since } \begin{matrix}\text{DB}\\ \text{DC}\end{matrix} : \text{BC} = \begin{matrix}\square\text{ on BA}\\ \square\text{ on CA,}\end{matrix} : \square\text{ on BC}$$

Therefore, 24, V. DB + DC : BC = $\square$ on BA + $\square$ on CA : $\square$ on BC.

Sch. 1. Proposition 31, bk. VI, is very comprehensive and renders Prop. 47, bk. I only a particular case. There is however a theorem still more general; it is given in his *Mathematical Collections* by Pappus, one of the later of the Greek Geometricians, who flourished at Alexandria during the reign of Theodosius, AD. 379—395 ;

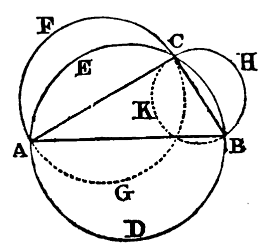
"If any ▱s AF, BE, be described on two sides, AC, BC, of any △ACB; and if the sides of the ▱s, DE, GF, be produced to meet, as in H, and if that point of intersection H and the vertex C of the triangle be joined, and the line HC produced to N; then these ▱s AF, BE are equal in area to a ▱AL, described on the .base AB, and having two of its sides AK, BL parallel to CN the line produced through the point of intersection H and the vertex C. and limited by the sides, DE, GF, of the two ▱s.



D. 1	C. 33, I.	∵ fig AKHC is a ▱, ∴ AK = CH ;
2	C. 31, I.	& ∴ fig. ABLK is a ▱, ∴ AKL = LB ;
3	Ax. 1, I. C.	∴ LB = CH ; & AL is a
4	35, I.	Now ▱NL = ▱BH = ▱CD ;
5	35, I.	& ▱NK = ▱AH = ▱AF ;
6	Ax 2, I.	∴ ▱NL + ▱NK, *i. e.*, ▱AL = ▱CD + ▱AF
7	Rec.	∵ *If any ▱s be described on the two sides*; *&c.* Q. E. D.

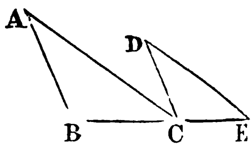
N.B. When ∠ABC is a rt. ∠, the s CD, AF & AL become squares, and the 47, I occurs, that the square on the hypotenuse is eoual to the squares on the sides including the rt. angle.

2. Circles, as well as squares, are similar figures, and it described with the sides, AB, AC, BC, of a rt ∠d △ for their diameters, then the circle with the hypotenuse AB for its diameter is equal in Area to the two circles, that have AC, BC, the sides about the rt. ∠, for their diameters. See Cor, 5 & 6 Pr. 20, VI.



PROP. 32.—THEOR.

If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another; the remaining sides shall be in a st. line.



E. 1	Hyp. 1.	Let $\triangle$s ABC, DCE have BA : AC = CD : DE;
2	,, 2.	and let AB be $\parallel$ DC and AC $\parallel$ DE;
3	Conc.	Then BC, CE form one st. line BE.
D. 1	H.	$\because$ AB $\parallel$ DC and AC meets them,
2	29, I.	$\therefore \angle$ BAC = $\angle$ ACD.
3	*Sim.* Ax. 1, I.	So $\angle$ CDE = $\angle$ ACD, $\therefore \angle$ BAC = $\angle$ CDE.
4	D. 3. & H.	And $\because$ $\triangle$s ABC, DCE have $\angle$ A = $\angle$ D, and BA : AC = CD : DE;
5	6, VI.	$\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DCE, and $\angle$B = $\angle$ DCE;
6	D. 2. Ax. 2, I.	and $\because$ $\angle$A = $\angle$ ACD, $\therefore$ whole $\angle$ ACE = $\angle$ ABC + $\angle$BAC:
7	*Add.*	add $\angle$ACB; then $\angle$s ACE+ACB = $\angle$s A+B+ACB.
8	32, I.	But $\angle$s A + B + ACB = 2 rt. $\angle$s;
9		$\therefore$ $\angle$s ACE + ACB = 2 rt. $\angle$s;
10	D. 8, 9.	and $\because$ at C in AC, the lines BC and CE on opp. sides of C make adj. $\angle$s = 2 rt. $\angle$s.
11	14, I.	$\therefore$ the lines BC, CE are in one st. line BE.
12	Rec.	Therefore, *if two triangles which have two sides &c.* Q. E. D.

SCH. The position of the given sides AC, DC, which are not homologous should be such as to form an angle, ACD, at the point of junction C; otherwise BC and CE may not be in one and the same st. line BE.

Prop. 33.—Theor.

In equal circles angles, whether at the centres, or circumferences have the same ratio which the circumferences, on which they stand, have to one another; so also have the sectors

Con. 1, IV. In a given $\odot$ to place a st. line equal to a given st. line which is not greater than the diam of the $\odot$. Pst. 1, I.

Dem. 28, III. In eq. $\odot$s, eq. st. lines cut off eq. arcs.

27, III. In eq. $\odot$s the $\angle$s on eq. arcs are equal, whether they be at the centres or $\odot$ces.

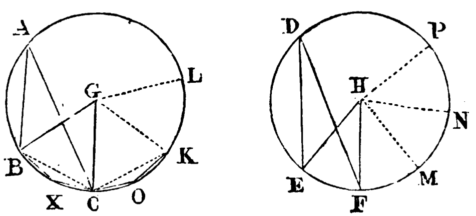
Def. 5, V. 20, III. The $\angle$ at the cen. is double of the $\angle$ at the $\odot$ce upon the same base.

15, V. Magnitudes have the same R. to one another which their equimultiples have.

4, I. Def. 11, III. Sim. segs. of $\odot$s are those in which the $\angle$s are equal.

24, III. Sim. segs. of $\odot$s upon eq. st. lines are eq. to one another. Ax. 3, I.

E. 1	Hyp. 1.	Let $\odot$ ABC $=$ $\odot$ DEF, having G & H as centres.
2	„ 2.	1°. On arcs BC, EF let there be at the centres $\angle$s BGC EHF, and at the $\odot$ces $\angle$s BAC, EDF;
3	Conc. 1.	then arc BC : arc EF $=$ $\angle$ BGC : $\angle$ EHF, and $\angle$ BAC : $\angle$ EDF.
4	Hyp3.	2°. Also on arcs BC, EF let there be sectors, BGC and EHF;
5	Conc.	then arc BC : arc EF $=$ sect. BGC : sect. EHF.



CASE I, *Of angles at the centres or $\odot$ces of equal circles.*

C. 1	1, IV.	In $\odot$ ABC take any eq. arcs BC = CK = KL,
2		and in $\odot$ DE any eq. arcs, EF = FM = MN;
3	Pst. 1, I.	and join GK, GL, HM, HN.
D. 1	28, III.	$\because$ arcs BC = CK = KL,
	27, III.	$\therefore$ $\angle$s BGC = CGK = KGL;
2		$\therefore$ the mult. which arc BL is of arc BC, that same mult. is $\angle$ BGL of BGC.
3	*Sim.*	So the mult. which arc EN is of arc EF, that same mult. is $\angle$ EHN of $\angle$ EHF;
4	27, III.	and if arc BL = > or < arc EN, $\angle$ BGL = > or < $\angle$ EHN.
5	H. 1.	Now $\because$ there are 4 Ms, arcs BC, EF, and $\angle$s BGC and EHF;
6	D. 2.	and of arc BC and $\angle$ BGC equims, arc BL and $\angle$ BGL are taken;
7	D. 3.	and of arc EF and $\angle$ EHF equims, arc EN and $\angle$ EHN;
8	D. 4.	and $\because$ if arc BL > = or < arc EN, $\angle$ BGL > = < $\angle$ EHN;
9	Def. 5, V.	$\therefore$ arc BC : arc EF = $\angle$ BGC : $\angle$ EHF.
10	20, III.	and $\because$ $\angle$ BGC = 2 $\angle$ BAC, and $\angle$ EHF = 2 $\angle$ EDF,
11	15, V.	$\therefore$ $\angle$ BGC : $\angle$ EHF = $\angle$ BAC : $\angle$ EDF;
12	Conc.	$\therefore$ arc BC : arc EF = $\angle$ BGC : $\angle$ EHF = $\angle$ BAC : $\angle$ EDF.

CASE II. *Of angles at the vertex of equal sectors.*

C. 1	Pst. 1, I.	Join BC, CK, and in arcs BC, CK take any $\cdot$s X, O;
2	Pst. 1, I.	and join BX, XC, CO, OK,
D. 1	H.	$\because$ in $\triangle$ GBC, GCK, BG, GC = CG, GK, and $\angle$ BGC = $\angle$ CGK.
2	4, I.	$\therefore$ base BC = base CK, and $\triangle$ GBC = $\triangle$ GCK.

3	C.1.Ax.3, I.	And arc BC = arc CK, ∴ rem. arc BLC = rem. arc CLK;
4	27, III, Def. 11, III.	∴ ∠ X = ∠ O, and seg. BXC is sim. to seg. CKO.
5	D. 2.	and ∵ BC = CK,
	24, III.	∴ seg. BXC = seg. CKO.
6	D. 2.	And △ BGC = △ CGK, ∴ the whole sect. BGC = the whole sect. CGK.
7	*Sim.*	So sectors KGL = BGC = CGK,
8	*Sim.*	and sectors EHF = FHM = MHN;
9	Conc.	∴ the mult. which arc BL is of arc BC, that same mult. sect. BGL is of sect BGC;
10	*Sim.*	and the mult. which arc EN is of arc EF, that mult. sect. EHN is of sect EHF;
11		and if arc BL = > or < arc EN, the sect. BGL = > or < sect. EHN.
12	Hyp.	Now ∵ there are 4 Ms, the arcs, BC, EF, and the sects. BGC, EHF;
13	D. 9.	and of arc BC and sect. BGC, arc BL and sect. BGL are equims.;
14	D. 10.	and of arc EF and sect. EHF, arc EN and sect. EHN are equims.;
15	D. 11.	and ∵ if arc BL > = or < arc EN, the sect. BGL > = or < sect. EHN;
16	Def. 5, V.	∴ arc BC : arc EF = sect. BGC : sect, EHF.
17	Rec.	∵ *In equal circles, angles, &c.*

Q. E. D.

Cor. 1. *The sectors are to each other as their angles.*

For, if arc BC : arc EF = ∠ BGC : ∠ EHF;
and arc BC: arc EF = sect. BGC : sect. EHF,
then 11, V. sect. BGC : sect EHF = ∠ BGC : ∠ EHF.

Cor. II. *Similar sectors of the same or equal circles are equal.*

Cor. III. *An angle at the centre of a circle is to four rt. angles as the arc on which it stands to the circumference of the circle.*

For, the $\angle$ at the centre : one rt. $\angle$ = arc subtending central $\angle$: arc subtending a rt. $\angle$ or quadrant;

Then, 4, V. $\angle$ at cen. : 4 rt. $\angle$s = arc of central $\angle$: whole $\odot$ce.

COR. IV. *In different circles the arcs of equal angles at the centres or circumferences are similar.*

COR. V. Hence, *similar segments are contained by similar arcs* and *vice versâ.*

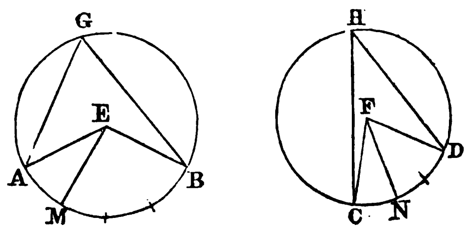
SCH. 1. If the arcs and sector had all been in one circle the proof would have been the same,—for H would have coincided with G, and D,E,F,M,N would have been points in the circumference of circle ABL.

2. The second part of the Proposition was added by THEON, the Ptolemaist, and father of the renowned but unfortunate HYPATIÁ, of Alexandria, in the time of the elder Theodosius;—it is given in the Commentary on PTOLEMY'S *Almagest.*

3. That the angles at the circumferences are as the arcs on which they stand, follows also as a Corollary from Prop. 20, bk. III.

Prop. 33 . *a.* *In the same or in equal circles AGB, CHD angles, whether at the centres, as $\angle$s AEB, CFD, or at the circumferences, $\angle$s AGB, CHD, have the same ratio as the arcs, AB, CD. on which they stand.*

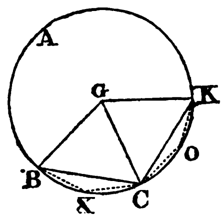
E & C. 1	Def. 1, V.	Let AM or CN be a com. meas. of arcs AB, CD;
2		& arc AMB = 4 arc AM, and arc CND = 3 CN;
3	Pst. 1, I.	then join EM and FN.



D.	1	27, III.	∵ $\angle$AEM = $\angle$CFN, & $\angle$AEB = 4 $\angle$AEM, & $\angle$CFD = 3 $\angle$CFN;
	2	7 *a* V.	∴ $\angle$AEB : $\angle$CFD = 4 : 3; *i. e.*, arc AMB: arc CND = 4 : 3;
	3	11, V.	∴ $\angle$AEB : $\angle$ CFD = arc AMB : arc CND.
	4	20, III.	And ∵ $\angle$s AEB, CFD are mults. of $\angle$s G & H by 2,
	5	15, V.	∴ $\angle$G : $\angle$H = arc AMB : arc CND Q. E. D.

Cor. Since the $\angle$s and arcs are proportional when commensurable, ∴ *they are also proportional when incommensurable.* See Sch. 3—5 & Use 1, 16, VI.

Prop. 33 *b*. *In the same circle, ABK, or in equal circles, the sectors BGC, GCK, that stand on equal arcs, BC, CK, are equal.*



C.	1		Take any . s X,O, in the arcs BC, CK,
	2	Pst. 1, I.	& join BX, XC, CO, OK.
D.	1	*Sub.*	From $\odot$ ABK take separately arcs BC & CK,
	2	Ax. 3, I.	∴ rem. arc. BAC = rem. arc CAK;
	3	27, III.	∴ $\angle$BXC is sim. to $\angle$COK, and seg. BXC to seg. CKO.
	4	27, III.	Again ∵ BG = GC = GK, & $\angle$ BGC = $\angle$CGK.
	5	4, I.	∴ $\triangle$ BGC = $\triangle$CGK, & base BC = base CK.
	6	D. 3. 5.	And ∵ seg. BXC is sim. to seg. CKO, & their bases, BC, CK equal.
	7	24, III.	∴ seg. BXC = seg CKO, & sect, BGC = sect CGK.
	8	*Sim*·	So, if sect. CGK were in a $\odot$ which is equal to $\odot$ ABC, sect. CGK = sect. BGC.

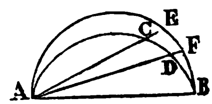
Prop 33 *c*. *In the same circle ABK*, as in fig. to 33 *b. or in equal circles,* ABG, CHD, as in fig. to 33 *a, sectors* AEB, CFD, *have the same ratio as the arcs,* AMB, CND, *on which they stand.*

E. & C.	1	Fig. 33, *a*.	Let arc AM = arc CN, & be a com. measure of arcs AB, CD;
	2		& let arc AB = 4 arc AM, & arc CD = 3 arc CN,
	3	Pst. 1, I.	then join EM and FN.
D.	1	33, *b*. VI.	∵ sect. AEM = sect CFN;
	2	C. 2.	& ∵ sect. AEB = 4 sect. AEM, & sect. CFD = 3 sect. CFN;
	3	7 *a* V.	∴ sect. AEB : sect. CFD = 4 : 3; and arc AB : arc CD = 4 : 3;
	4	11, V.	∴ sect. AEB : sect. CFD = arc AB : arc CD.

Cor. Since the arcs and sectors are thus proportional when commensurable, ∴ *they must also be proportional when incommensurable.*

4. From this 33rd Proposition it results, that the angle at the centre of a circle is said to be measured by the arc on which it stands.

Use and Appl. 1. *If arcs* ACB, AEB, *of different circles have a common chord,* AB, *the lines,* AC, AD, *diverging from one of its extremities,* A, *will cut the arcs proportionally,* i. e. BF : FE = BD : DC.



D. 1	Ax. 8, I.	∵ $\angle$s BAF, BAD are identical, and also $\angle$s EAF, CAD ;
2	33, VI.	∴ arc BF : arc FE = $\angle$ BAF : $\angle$EAF,
3	„ „	also arc BD : arc DC = $\angle$BAF : $\angle$EAF ;
4	11, VI.	∴ arc BF : arc FE = arc BD : arc DC.

Q. E. D.

2. *The arcs,* A, A′ *of unequal circles are in a ratio compounded of their central angles* a, a' *and their radii,* R, R′.

C. 1		With rad. = R desc. an $\angle = a'$,
2		and let the subtending arc be m.
D. 1	H. & C. 2	∵ arcs A and m have an eq. R, ∴ $A : m = a : a'$;
2	C. & H.	and ∵ m and A′ have eq. central $\angle$s ;
3	Cor. 5, 20, VI.	∴ $m : A', = R : R'$
4	D. 1 & 2.	But A : A′ is compounded of $\left\{\begin{matrix} A : m \\ m : A', \end{matrix}\right.$ or of the equivalent, ratios, $a : a'$ and R : R′.

3. *Central angles* a, a', *are in a ratio compounded of the direct ratio of their arcs* A, A′, *and the inverse ratio of their radii* R, R′.

D. 1	Use 2, 33, VI.	For $A : A' = \left\{\begin{matrix} a : a', \\ R : R' \end{matrix}\right.$
2		Let each of the equal ratios be compounded with R′ : R;
3	D. 1 and 2	then $\left\{\begin{matrix} A : A' \\ R' : R \end{matrix}\right\} = \left\{\begin{matrix} a : a' \\ R : R' \\ R' : R \end{matrix}\right.$
4	Sch. 1, 16, VI.	But $\left.\begin{matrix} R : R' \\ R' : R \end{matrix}\right\}$ is a ratio of equality,
5	Conc.	∴ $\left.\begin{matrix} A : A' \\ R' : R \end{matrix}\right\} = a : a'$

See Lardner's *Euclid*, p. 224.

OBS. "And herewith," remarks Captain Thomas Rudd, Chiefe Engineer to Charles I., "is the first six Books of EUCLIDE ended. There be hereafter added certain Propositions, which although they be not EUCLIDES, yet because they are both witty and usefull, I thought it good not to omit." RUDD's *Euclides Elements*, A.D. 1651, p. 253.

SUBSIDIARY PROPOSITIONS.

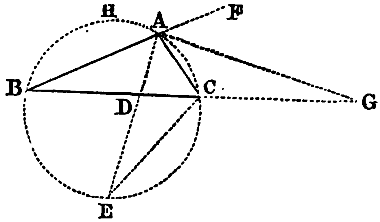
PROP. B.—THEOR.

If an angle of a triangle be bisected by a st. line which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the st. line which bisects the angle.

CON. 5, IV. To desc. a $\odot$ about a given $\triangle$. Pst 2, 1, I.

DEM. 21, III. The $\angle$s in the same seg. of a $\odot$ are equal.
3, II. If a st. line be divided into any two pts, the ▭ contained by the whole and one of the pts is eq. to the ▭ contained by the two parts together with the square of the aforesaid part.
35, III. If two st. lines cut one another within a $\odot$ the rect. contained by the segs. of one is eq. to the rect. contained by the segs. of the other.
32, I. 4, VI. 16, VI.

E.	1	Hyp.	In $\triangle$ ABC let $\angle$ A be bisected by AD;
	2	Conc.	then BA . AC = BD . DC + AD^2.
C.	1	5, IV.	About $\triangle$ ABC describe $\odot$ ACB;
	2	Pst. 2 1, I.	prod. AD to $\odot$ce E, and join EC;



D. 1	H. 21, III.	Then $\because \angle BAD = \angle CAE$, & $\angle ABD = \angle AEC$;
2	32, I. 4, VI.	$\therefore \triangle ABD$ is eq. ang. with $\triangle AEC$, & $\therefore BA : AD = EA : AC$;
3	16, VI. 3, I.	& $\therefore BA . AC = EA . AD$, *i. e.*, $= ED . DA + AD^2$.
4	35, III.	But $ED . DA = BD . DC$.
5	Conc.	$\therefore AB . AC = BD . DC + AD^2$.
6	Rec.	$\therefore$ *If an angle of a triangle, &c.* Q. E. D.

Cor. From the hypothesis, that the exterior vertical angle, CAF, is bisected by GA, which also cuts the base produced BCG in G and the circle in H, the conclusion follows, that *the rectangle of the sides together with the square of the line which bisects the exterior angle is equal to the rectangle of the whole line produced and the extreme segment*; i. e $BA . AC + AG^2 = BG . GC$.

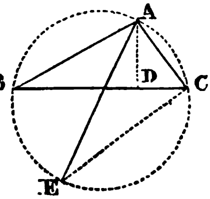
Sch. Prop B and its Corollary combined, may be thus enunciated; "if the vertical or ext. vertical $\angle$ of a $\triangle$ be bisected by a st. line, which cuts the base, or the base produced, the square of the st. line shall be equal to the difference of the rectangles under the two sides and under the segments of the base or of the base produced."

In the case of the bisection of the vert. $\angle BAC$, $AD^2 = BA . AC - BD . DC$.
And " ext. vert. $\angle CAF$, $AG^2 = BG . GC \sim BA . AC$

Prop. C.—Theor.

If from any angle of a triangle a st. line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

CON. 5, IV. DEM. 31, III. 21, III. 4, VI. 16, VI.



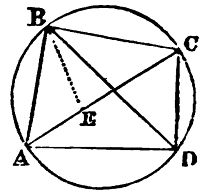
E. 1	Hyp.	In △ ABC, from ∠ A let AD be ⊥ base BC,
2	Conc.	then BA . AC = AD . AE.
C.	4, VI.	About △ ABC desc. ⊙ ACB and its diam. AE, & join EC.
D. 1	31, III. & 21 III.	∵ rt. ∠ BDA = ∠ ECA, & ∠ ABD = ∠ AEC;
2	4, VI.	∴ △ BDA is eq. ang. with △ ACE; & ∴ BA : AD = EA : AC;
3	16, VI.	∴ BA . AC = EA . AD.
4	Rec.	*If therefore from any angle, &c.* Q. E. D.

COR.—*If two* △*s*, ABC, ACE, *be inscribed in the same or in equal* ⊙*s*, *the rect. under the two sides of the one*, BA . AC, *shall be to the rect. under the two sides of the other*, EC . CA, *as the perp.* AD, *which is drawn from the vertex* A, *to the base*, BC, *of the one*, *is to the perp. which is drawn from the vertex* C *to the base*, AE, *of the other :* i. e. BA . AC : EC . CA = AD : *perp. from C.*

PROP. D.—THEOR.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

CON. 23, I. DEM. 21, III. 4, VI. 16, VI. 1, II. If there be two st. lines, one of which is divided into any number of pts.; the rect. contained by the two st. lines is equal to the rectangles contained by the undiv. line and the several pts. of the divided line.



E. 1	Hyp.	In a ⊙ let a qu. lat. ABCD be insc with diags. AC, BD;
2	Conc.	then AC . BD = AB. CD + AD . BC.
C.	23, I.	Make ∠ ABE = ∠ DBC.
D. 1	*Add.*	Add ∠ EBD to the equals, ∴ ∠ ABD = ∠ EBC.
2	21, III.	and ∵ ∠ BDA = ∠ BCE ∴ △ ABD is eq. ang. with △ EBC;
3	4, VI.	∴ BC : CE = BD : DA,
	16, VI.	and ∴ BC . DA = BD . CE.
4	C. 21, III	Again, ∵ ∠ ABE = ∠ DBC, and ∠ BAE = ∠ BDC,
5	4, VI.	∴ △ ABE is eq. ang with △ BCD, ∴ BA : AE = BD : DC;
6	16, VI.	∴ BA . DC = BD . AE.
7	D. 3.	But BC . DA = BD . CE;
8	1, II.	∴ the whole ▭ AC . BD = AB . CD + AD.BC.
9	Rec.	Therefore, *the rectangle contained, &c.*

Q. E. D.

Sch. This Proposition is named Ptolemy's Theorem, and is a Lemma at p. 9 of his Almagest, or Μεγαλη Συνταξις of which it is said, "Ptolemy appears as a splendid mathematician, and an (at least) indifferent observer."

It is curious to note how editor after editor of Euclid has followed the identical diagrams of the earliest printed editions. Ptolemy's thirteen books "*Magnæ Constructionis, Id est, Perfectæ cœlestium motuum, pertractationis,*" published at Basle by Symon Grynoeus in the year 1538, contain several diagrams, and the one to Prop. D, bk. VI, among the number, which have scarcely undergone any change since that time. *Billingsley's* English Euclid also furnishes very many examples from which to prove the *imitativeness* of succeeding Editors, and by which to justify, if need be, the continuation of the practice.

PROP. E.—THEOR.

The diagonals, AB, CD, *of a quadrilateral* ACBD, *inscribed in a circle,* ABC *are to one another as the sums of the rectangles under the sides adjacent to the extremities of those diagonals :* i. e., AB : CD : = CA . AD + CB . BD : AC . CB + AD . DB.

CON. 12, I. DEM. Cor. C, VI. Sch. 3, 24, V. In any number of magnitudes of the same kind forming two series, if the ratios of the 1st to the 2nd, of the 2nd to the 3rd, of the 3rd to the 4th, and so on, be the same in the two series ; then any two combinations whatever, shall be to one another as two similar combinations of the corresponding magnitudes of the second series. 4, 5, VI. 11, V.

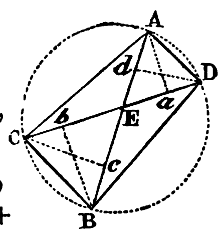
C. | *Sup.* | Let AB, CD, cut one another in the . E.

CASE I. *Suppose that AB cuts CD at rt. ∠s.*

D. 1 | Hyp. | ∵ △s ACD, BCD, CAB, DAB are in the same ⊙;

2 | Cor. C, VI | ∴ perps. AE, BE CE and DE are to one another as the ▭s AC . AD, BC . BD, CA . CB, DA . DB;

3 | Sch. 3, 24, V. | ∴ AE + EB, or AB : CE + ED, or CD, = AC . AD + BC . BD : CA . CB + DA . DB.



CASE II. Or, *Suppose that AB cuts CD* not *at rt. ∠s.*

C. | 12, I. | Draw the perps. A*a*, B*b*, C*c*, D*d*.

D. 1 | *Sim.* | As before, A*a* + B*b* : C*c* + D*d* = AC . AD + BC . BD : CA . CB + DA . DB;

2 | C. | but ∵ △s AE*a*, BE*b*, CE*c*, and DE*d* are eq. ang.;

D. 3	4, 5, VI.	∴ Aa, Bb, Cc, Dd, are to one another, as AE, BE, CE and DE;
4		∴ Aa, + Bb : Cc, + Dd = AE + BE : CE + DE = AB : CD.
5	11, V.	∴ AB : CD = AC . AD + BC . BD : CA . CB + DA . DB.
6	Rec.	Therefore, *the diagonals of a quadrilateral, &c.* Q. E. D.

USE AND APP. By aid of the last two Propositions, D & E, bk. VI, the following Problem may be solved.

Given four st. lines, any three of which are together greater than the fourth, to construct a quadrilateral, of which the sides shall be equal to those four given st. lines, in a given order, each to each, and of which also its angular points lie in the circumference of a circle.

By E, VI. The ratio of the diagonals is given; they are as the sums of the rectangles under the sides;

By D, VI. The rectangle of the diagonals is given, being equal to the sum of the rectangles under the opp. sides;

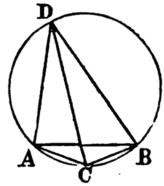
By Prob. 5 Use and App. 13, VI., we find the two st. lines; in this case, they are the two diagonals equal to a given rectangle;

And by 22, I, having now the sides and diagonals of the quadrilateral, we construct two triangles the sum of which will equal the required quadrilateral.

PROP. F.—THEOR.

If AB, *a segment of a circle,* ABD, *be bisected in* C, *and from the extremities,* A, B, *of the base of the segment, and from* C *the point of bisection, st. lines be dawn to any point* D *in the circumference; the sum of the two lines* AD + DB *drawn from the extremities of the base, will have to the line* DC *drawn from the point of bisection the same ratio which the base* BA, *of the segment has to* AC *the base of half the segment.*

DEM. D, VI. 1, II. 14, VI.



D. 1	H.	∵ ADBC is a qu. lat. inscribed in a ⊙, of which AB and CD are diagonals;
2	D. VI.	∴ AD . CB + DB . AC = AB . CD.
3	H.	but ∵ AC = CB, ∴ AD . CB = AD, AC
4	D, 2, 3.	∴ AD . AC + BD . AC = AB . CD.
5	1, II.	But AD . AC, & BD . AC are the rectangles contained by AC & AD + DB;
6		∴ rect. AC . (AD + DB) = rect. AB C. D;
7	14, VI.	& ∵ the sides of eq. rectangles are reciprocally propl;
8	Conc.	∴ AD + DB : DC = AB : AC.
9	Rec.	Therefore, *if a segment of a circle be bisected &c.* Q. E. D.

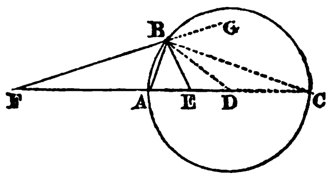
SCH. This and the following Subsidiary Propositions have been adopted, with some slight alterations from BELL's *Plane Geometry*, p. 194 — 198.

PROP G.—THEOR.

If two points, E & F, *be taken in the diameter* AC *of a circle, or of the diameter produced,* CF, *such that the rectangle,* ED.DF, *contained by the segments intercepted between them and* D *the centre of the circle be equal to the square of* AD *the semidiameter*; *and if from these points two st. lines, EB, FB, be inflected to any point whatever,* B, *in the circumferenece of the circle, the ratio of the lines inflected,* FB : BE, *will be the same with the ratio of the segments,* FA : AE, *intercepted between the two first mentioned points and the circumference of the circle.*

DEM. 17, VI. 6, VI. 4, VI. 16, V. *altern.* 17, V, *dividendo.* 11, V.

C | Pst. 1 & 2, I. | Join DB, CB, AB & prod. FB to G.



D. 1	H.	∵ $FD \cdot DE = AD^2 = DB^2$;
2	17, VI.	∵ $FD : DB = DB : DE$;
3	D. 2.	& ∵ in △s FDB, BDE the sides about the common ∠D are propl.;
4	6, VI.	∴ △ FDB is eq. ang. with △ BDE, ∠DBF = ∠DEB, & DBE = ∠DFB.
5	4, VI.	∴ $FB : BD = BE : ED$,
6	16, V.	& *alt.* $FB : BE = BD : ED$, or $AD : DE$.
7	D, 2.	But ∵ $FD : DB$ or $DA = DA : DE$;
8	17, V.	∴ *div.* $FA : DA = AE : ED$.
9	16, V.	& *alt.* $FA : AE = DA : ED$.
10	D. 6.	Now $FB : BE = AD : DE$,
11	11, V.	∴ $FB : BE = FA : AE$.
12	Rec.	∴ *If two points be taken in the diameter, &c.* Q. E. D.

COR. 1. ∵ $FB : BE = FA : AE$, ∴ *on joining* AB, *by* 3, VI., ∠FBE *is bisected by* AB.

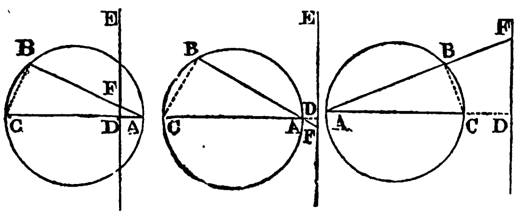
COR. 2. *Also on joining* BC, *the ext. vert.* ∠EBG *is bisected by* BC; for

D. 1	17, VI. & 18, V.	∵ $FD : DB$ or $DC = DC : DE$, ∴ *comp.* $FC : DC = CE : ED$;
2	17, V.	& ∵ $FA : AD$ or $DC = AE : ED$, ∴ *ex. æq.* $FA : AE = FC : CE$.
3	G, VI.	But $FB : BE = FA : AE$, ∴ $FB : BE = FC : CE$;
4	A, VI.	∴ ext, ∠EBG is bisected by BC. Q. E. D.

Prop. H.—Theor.

If from one extremity, A, *of* AC *the diameter of a circle* ABC, *a chord* AB *be drawn, and a perpendicular* DE, *to the diameter, cut both the diameter and the chord either internally or externally in* D *and* F, *the rectangle,* CA . AD, *under the diameter and its segment reckoned from that extremity* A, *is equal to the rectangle* BA . AF, *under the chord and its corresponding segment.*

Dem. 1, III. 15, I. 4, VI. 16, VI.



D.	1	H. 31, III.	∵ ∠ABC is in a semic, ∴ ∠ABC is a rt. ∠;
	2	H. 15, I.	but ADF is also a rt. ∠, & ∠BAC = ∠DAF;
	3		∴ △ABC is eq. ang. with △ADF;
	4	4, VI.	∴ BA : AC = AD : AF:
	5	16, VI.	∴ BA . AF = AC . AD.
	6	Rec.	∵ *If from one extremity of the diameter, &c.*

Q. E. D.

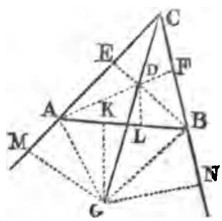
Prop. K.—Theor.

If the angles, A *&* B, *at the base* AB *of a triangle* ABC, *be bisected by two lines,* AD, BD, *that meet, as in* D, *and the exterior angles at the base, formed by producing the two sides* CA,CB, *be similarly bisected by* AG & BG*; then the two points of concourse,* D & G, *and the vertex,* C, *shall be in one st. line, which shall bisect the vertical angle,* ACB.

Preliminary Theorem, that may be demonstrated by *superposition*, "If two $\triangle$s have two sides of the one respectively equal to two sides of the other, and the $\angle$ opp. one of the sides in the first equal to the $\angle$ opp. to the equal side in the second, these $\triangle$s are equal when they are of the same species or affection, *i. e.*, when they are both acute-angled, both right-angled, or both obtuse-angled. See Sch. P, 7, VI.

Con. 12, I. Dem. 26, I. & Prel. Theor.

C.	1	12, I.	Draw DE, DF, DL $\perp$ to the sides AC, BC & AB;
	2	"	& GM, GN $\perp$ to the sides produced AM, BN;
	3	"	& GK $\perp$ to the side AB.



D.	1	Hyp.	$\because$ in $\triangle$s ADE, ADL, $\angle$EAD = $\angle$DAL, $\angle$s at E & L also equal, & AD common.
	2	26, I.	$\therefore$ AL = AE & DL = DE.
	3	*Sim.*	So BL = BF, & DL = DF;
	4	"	also AM = AK, GM = GK, BN = BK, & GN = GK.
	5	D, 2, 3.	& $\because$ DE & DF each = DL, $\therefore$ DE = DF, & *sim.* GM = GN
	6	D. 5, C.	Again, $\because$ in $\triangle$s CED, CFD, CD, DE = CD, DF, & $\angle$s at E & F are rt. $\angle$s;
	7	*Prel. Theor.*	$\therefore$ $\triangle$ CED = $\triangle$CFD, & $\angle$ECD = $\angle$FCD.
	8	*Sim.*	So $\triangle$CMG = $\triangle$CNG, & $\angle$MCG = NCG;
	9	Conc.	$\therefore$ the lines CG & CD coincide.
	10	Rec.	$\therefore$ *If the angles at the base, &c.* Q.E.D.

Prop. L.—Theor.

In a triangle ACB, *as in the last proposition, the segments,* CM *or* CN, *of each side produced that are intercepted between the vertex,* C *and the external perpendiculars,* GM, GN, *are each equal to* S, *the semiperimeter of the triangle; the segments,* CF *or* CE, *of these sides next the vertex,* C, *are equal to* S—AB, *the excess of the semiperimeter above the base* AB; *and the segment* AE *or* BF, *of each of these sides next the base is respectively equal to* S—BC, *or to* S—AC, *the excess of the semiperimeter above the other side.*

Dem. 26, I. Ax. 6 & 7, I. Ax. 2, I.

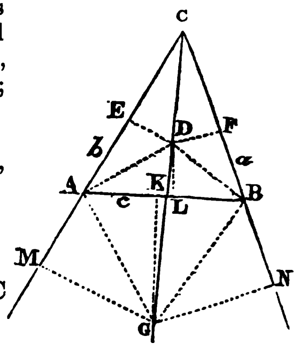
D. 1	26, 1.	∵ AK = AM, & BK = BN, ∴ CM + CN = perimeter;
2	26, I.	& CM = CN = S, the semiperimeter, & AM = S — AC.
3	Ax. 6, I.	Also 2 CE + 2 AE + 2 AM = perimeter = 2 CF + 2 AL + 2 BL.
4	D. 5, K. D. 2, K	Now CE = CF, & AE = AL;
5	Ax. 6 & 7, I.	∴ 2 AM = 2 BL, & AM or AK = BL;
6	*Add.* Ax. 2.	adding KL to both, ∴ AL = BK.
7	D. 5 & 4.	And ∵ AM = BL, & AE = AL,
8	Conc. 1	∴ ME = AB, & CE = S — AB, *i. e.*, CE = excess of semiperimeter above the base.
9	D. 7, 6 & K, 4	And ∵ AE = AL = BK = BN;
10	Conc. 2.	∴ AE = S — BC;
11	D. 7, & K 3. D. 2	& ∴ AM = AK = BL = BF, & AM = S — AC,
12	Conc. 3.	∴ BF = S — AC *i. e.*, each seg. of the side next the base is equal to the excess of the semiperimeter above the other side.
13	Rec.	∴ *the segment of each side, &c.* Q. E. D.

Prop. M.—Theor.

The area of a triangle is a mean proportional between two rectangles, the sides of one of which are equal to the semiperimeter and its excess above the base, and the sides of the other equal to the excessess of the semiperimeter above the other two sides.

Dem. K & L, VI. 4, VI. 29, I. Cor. 1, 23, VI. 41, I. 13, I. 32 .I 16, VI.

E. 1	Hyp. 1	Let ABC be a $\triangle$, of which in the sides CA & CB produced CM = CN = S, the Semiperimeter;
2	„ 2	S—AB = CE; S—AC = AM & S—BC = AE, *i.e.* the excesses of S above AB, AC & BC;
3	Conc.	then CM . CE : $\triangle$ABC = $\triangle$ ABC : AM . AE.
C	K & L, VI.	The fig. being constructed as in Pr. K & L, VI.
D. 1	29, I. 4, VI.	$\because$ ED $\parallel$ MG, $\therefore$ CE : ED = CM : MG;
2		but CM : CM = ED : ED;
3	Cor 1, 23, VI.	$\therefore$ CE . CM : ED . CM = CM . ED : MG . ED.
4	C.	But $\triangle$ ABC = $\triangle$s ADC, ADB, & CDB;
5	41, I.	& these $\triangle$s = $\frac{AC.DE}{2} + \frac{AB.DL}{2} + \frac{BC.DF}{2}$ = ED . CM.
6	13, I. 32, I.	And $\because$ $\angle$s EAL + MAK = 2 rt. $\angle$s, $\therefore$ $\angle$s EAD + MAG = 1 rt. $\angle$ = $\angle$s AGM + MAG;
7	Ax. 3, I.	$\therefore$ $\angle$ EAD = $\angle$ AGM;
8	C.	& $\because$ $\angle$s at E & M are. rt. $\angle$s, $\therefore$ $\triangle$s AGM, EAD are eq. ang.;



9	4, VI. 16, VI.	$\therefore$ AE : ED = MG : AM, & $\therefore$ AE . AM = MG . ED.
10	D 3.	But CM . CE : ED . CM = CM . ED : MG . ED becomes CM . CE : $\triangle$ ABC = $\triangle$ ABC : AE . AM;
11	H. 1, 2.	$\therefore$ S (S–AB) : ABC = ABC : (S–AC) (S—BC) Q. E. D.

Cor. Let the sides opp. $\angle$s A, B, C, be denoted by a, b, & c; then

$$s\,(s-c) : \triangle\ ABC = \triangle\ ABC : (s-a)\,(s-b);$$
$$\therefore \text{Area of } \triangle\ ABC = \sqrt{s\,(s-c)\,(s-a)\,(s-b)}.$$

Use and App. From this Prop. the Solution is obtained of the Problem *Given the three sides of a triangle to find the Area*; for, as in Cor. M. VI, the continual product of the Semiperimeter into the excesses of the semiper. above the three sides is equal to the square of the Area; whence the extraction of the square root gives the Area.

Ex. The sides of a triangle, ABC, are AB 221, BC 255 & AC 238 feet; required the Area.

Here $221 + 255 + 238 \times \frac{1}{2} = 357$ the Semiperimeter.

And $357-221 = 136$; $357-255 = 102$, & $357-238 = 119$, the Excesses.

Then Area$^2 = 357 \times 136 \times 102 \times 119 = 589324176$.

$\therefore$ Area of given $\triangle = \sqrt{589324176} = 24276$ square feet.

REMARKS ON BOOK VI.

1. In a general way it may be said that the Sixth Book, being an application of the Theory of Proportion propounded in the Fifth, treats chiefly of *similar* rectilineal and curvilinear figures, or of figures that differ in size, but not in form.

2. The Book contains 33 propositions by EUCLID, of which 10 are Problems, and 23 Theorems; to these have been added 13 Subsidiary Theorems.

3. The 11th, 12th, 13th, 18th and 25th Propositions are the most important among the Problems; and the 4th, 5th, 8th, 16th, 19th, and 31st, among the theorems.

4. As an *approximate* Classification for the Sixth Book, it may be divided, or rather arranged;

1°.—Into Propositions which treat of the Proportion existing between the *sides* of triangles; as Prop. 2, 4, 5, 6, 7, 8.

2°.—Into Propositions showing the Proportions between the *surfaces* of rectilineal figures; as Prop 1, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 26, 27 and 31.

3°.—Into those which relate to lines in Harmonical Progression; as Prop. 3 & A.

4°.—To the angles at the centres or circumferences of circles; as Pr. 33.

5°.—To Rectangles under the segments of chords; as Prop. B, C, D, E, F, G, H.

6°.—To certain Properties of the triangle when its angles are bisected, and its Semiperimeter compared with its sides; as Prop. K, L, & M.

5. In addition to the 10 Problems which EUCLID gives, there are several other very useful Problems evolved from the Principles established, as

1. To divide any rt. lined surface, Use P. 1.
2. To measure the height of an inaccessible object which casts an accessible shadow. Use 2, P. 2.
3. To divide a given line into proportional parts. Use 3, P. 2.
4.—To find a harmonic mean. Use 1, P. A. Use 2, P. 10.
5.—Eight Problems for ascertaining heights, distances, &c. Use P. 5.
6.—By aid of a square to measure inaccessible distances. Use 2, P. 8.
7.—To divide a given line into any number of equal parts. Use 1, P. 9.
8.—To divide a $\triangle$ into any number ef equal parts. Use 2, P. 9.
9.—To divide a given line in a given ratio. Use 1, P. 10.
10.—To find a third harmonical progressional. Use 3, P, 10.
11.—To construct a triangle, one side, the angle opposite to it, and the ratio of the other sides being given. Use 4, P. 10.

12.—To draw a line which, if two other lines were produced, would pass through their point of intersection. Use 5, P. 10.

13.—To continue a series of Ratios in progression. Use 1, P. 11.

14.—From the two first terms in a series to obtain the sum of the series. Use 3, P. 11.

15.—Problems relating to the Use and Application of the Sector. Use P, 12

16.—Problems relating to a mean Proportional. Use 1—6, P. 13.

17.—To construct an isosceles triangle equal to a given scalene triangle and with the same vertical angle. Use P. 15.

18.—Application of the Theory of Limits. Use P. 16.

19.—Application of the Theory of Proportion in Prop. 14, 15, 16 & 17. Use 1—4, P. 19.

20.—The practical methods of drawing Plans, Maps, &c. Use P. 18.

21.—Methods for ascertaining the relative areas of similar figures. Use 1 2, P. 19.

22.—To increase or diminish a rt. lined, or a circular figure in any ratio. Use 1, 3, P. 20.

23.—From the areas and corresponding sides of similar figures to deduce the part not given. Use 4, P. 23.

24.—To describe a rhombus equal to a given rectilineal figure, and with an angle equal to a given angle. Use P. 27.

25.—In a given triangle to inscribe the greatest possible parallelogram. having an angle equal to a given angle. Use P. 27.

26.—Several Problems for applying rt. lined figures to a given line. Use P. 29.

27.—To divide a line, so that the rectangle under the whole line and one part shall bear a given ratio, as $m : n$, to the square of the other part. Sch. 3, P. 30.

28.— Problems depending on a given st. line cut in extreme & mean ratio, Use P. 30.

29.—The construction of a quadrilateral under certain conditions. Use P. E.

30—Given the three sides of a triangle to find the Area. Use P. M.

6.—As there are several useful Problems, so are there many important Theorems referred to, or contained in the Notes to Bk. VI.

1.—The Theory of Transversal Lines. Sch. 2, P. 2.

2.—The Equation of a rt. line in Analytical Geometry. Cor. 2, P. 6.

3.—The Criteria of similarity in Triangles. Use P. 7.

4.—The General Principle, if from the vertex of a triangle two lines be drawn to the base making the angles at the base or their supplements each equal to the vert. angle then the triangle formed by those lines and the segments shall be similar to the whole triangle and to one another. Sch. 1, P. 8.

5.—Series of Magnitudes in continued proportion. Use, 2, P. 11.

6.—Parallelograms are equiangular when their areas and sides are reciprocally proportional. Sch. 1, P. 14.

7.—Reciprocal ratios, and ratios of equality, Sch. II, P. 16.

8.—The conversion of the equality of two ratios into the equality of two rectangles. Sch. III, P. 16.

9.—The leading Theorems for the Doctrine of Limits. Sch. IV, P. 16.

10.—Various deductions and processes stated in other words than the original propositions, as Use III, P. 17. Sch. P. 22. Sch. P. 28. Use 2°, P. 28. Use 2° and 3°, P. 29.

11.—The General Theorem respecting incommensurable Magnitudes. Sch. 2, P. 30.

12.—The General Theorem, by Pappus, under given conditions, of the equality in Area of a parallelogram or a circle on the base of a triangle to the sum of the parallelograms or circles on the other two sides. Sch. 1, 2, P. 31.

13.—Three Propositions substituted for P. 33, VI. Sch. 4, P. 33.

14.—The cutting of arcs proportionally. Use, 1, P. 33.

15.—The ratio of arcs in unequal circles, and of central angles: Use 2, 3, P. 33.

N. B.—Additional Corollaries will be found under their respective Propositions.

7.—The importance of the sixth Book to the Mathematician can scarcely be over-estimated; it is the head and crown of Plane Geometry. Whoever has mastered it, has added no little to his amount of useful knowledge, and consequently to the power of his mind. Whether or not he adopts the quaint Motto round the Effigies of an old English Geometer,—"Liefe is Deathe and Death is Liefe," certain it is that he has planted his feet on one

of the summits of Human Wisdom, and as he looks around, either to survey the Country already traversed, or to take note of the Heights still rising before him, he may inscribe upon his work, in the spirit of devotion;

Laus Deo,

I thank God, and take courage.

SO NOW—SO EVER.

EUCLID'S PLANE GEOMETRY PRACTICALLY APPLIED.

Synoptical INDEX to Books I—VI.

Part I. containing Bks. I.—II.

INTRODUCTION.

BOOK I.

The Geometry of Plane Triangles.

FOURTEEN PROBLEMS :

Prop. 1, 2, 3;—9, 10, 11, 12;—22, 23;—31;—42;—44, 45, 46.

45, 130. To descr. a ▱ = a given rectil. fig., and having an ∠ = a given ∠.

USE.—To measure the superficial content of any rectil. fig.; 2, To change any rectil. fig. into a △, and then into a rect. of eq. Area; and 3. To straighten a crooked boundary without changing the dimension.

46. 132. To describe a Square on a given st. L.

COR. 1. The squares on eq. Lines are eq.; and *conversely*.
2. Every parallelogram with one rt. ∠ has all its ∠s rt ∠s.

SCH.—Given the diagonal to construct a square.

USE.—The Geometrical Square; its construction; its use in ascertaining inaccessible distances, as heights.

THIRTY FOUR THEOREMS.—ONE LEMMA;

Prop. 4, 5, 6, 7, 8;—13, 14, 15, 16, 17, 18, 19, 20, 21;—24, 25, 26;—Lemma;—27, 28, 29, 30,—32, 33, 34, 35, 36, 37, 38, 39, 40, 41,; —43; 47, 48.

4. 52. *Important.*—If two △s have each two sides and their included ∠ eq. the bases and other ∠s are eq. and the △s equal.

SCH.—The equality is perfect, not in Area only. USE.—The first *criterion* for establishing the *equality* of △s; very frequently applied; and useful, along with the Theory of Representative Values, for ascertaining inaccessible distances.

5. 54. The ∠s at the base of an isosc △ are eq.; and if the eq. sides are produced, the ∠s on the other side also are equal.

COR. Every equal triangle is also equiangular.

6. 57. *Conversely.* If two ∠s of a △ are eq. the sides opp. the eq. ∠s also are equal.

COR. Every equiangular △ shall be equilateral.

SCH. 1, 2.—Converse Theorems not universally true. 3. Two modes of Demonstration, *direct* and *indirect*. USE. To determine the Height of an object by its shadow.

7. 59. On the same side of the same base there cannot be two △s, with sides terminated in one extremity equal, and also the sides equal terminated in the other extremity.

The Dilemma, or Double Antecedent, Prop. 7, used only to prove 8, I.

SCH.—Prop. 5, 6, 18 & 19, combined, prove, "One ∠ of a △ is, > =, or < another ∠, as the side opposed is >, =, or < the other side opposed; and "*vice versa.*" USE 1. The Perp. is the shortest L from a . to a given L. 2. From one . only two eq. lines to a given L. can be drawn. 3. All heavy bodies free to move seek the . nearest the earth's centre. 4. To construct a △, having the base, the less ∠ at the base, and the diff. of the sides given.

20. 81. Any two sides of a △ are together greater than the third side.

COR. The diff. of any two sides of a △ is less than the remaining side.

N.B.—More assumed in the Cor. than is expressed in Ax. 5.

USE 1.—Of all lines from one . to another and reflected to a third; those the shortest which make the ∠ of incidence = the ∠ of reflection. 2. Natural causes act by the shortest lines; hence, by means of a mirror to construct a △ of which the Perp. is representative of the height of an object.

21. 84. If from the ends of a side of a △ two Ls be drawn to a . within the △. these lines are < the other two sides, but contain a greater ∠.

Applied in Optics, Astronomy, and Architecture.

24. 91. If two △s have two sides of one = two sides of the other, but the ∠ contained by the two sides of the one > the ∠ contained by the two sides of the other, the base of that which has the gr. ∠ shall be > the base of the other.

25. 92. *Conversely.*—If two △s have two sides of one = two sides of the other, each to each, but the base of one > the base of the other, the ∠ opp. the gr. base shall be > the ∠ opp. the less base.

Pr. 4, 8, 24 and 25, may be combined; "If two △s have each two sides = two sides, the third side of the one will be >, <, or = the third side of the other, as the ∠ opposed in one is >, <, or = the ∠ opposed in the other; and *vice versâ.*

26. 93. *Important.*—If two △s have two ∠s and a side of the one = two ∠s and a side of the other; the other sides shall be eq. each to each, and the third ∠ of the one = the third ∠ of the other.

SCH.—The third *criterion* of the equality of △s. In two or more △s, any three parts of which one must be a side, being given equal, the equality of the other parts will follow.

Use.—To ascertain the perp. height of a mountain, as well as the distance from the base to the foot of the perp.

34. 110. The opp. sides and $\angle$s of ▱s are eq. to one another, and the diagonal bisects them; and *conversely*.

Sch.—If a quadril. fig. have any *two* of certain *ten* data, it will also have the others. By combining the ten, 360 questions are raised.

Use. 1.—The construction of the parallel ruler depends on this Prop. It is also useful, 2, to divide a line into any number of eq. parts ; 3, to construct the Sliding Scale, called the *Vernier* or *Nonius* for measuring *minute* parts ; 4, to obtain the distance between two objects ; 5, to continue a st. line when an obstacle intervenes ; 5, to divide a ▱ into two eq. pts. from a · in one of the sides, &c.

35. 113. Parallelograms on the same base and between the same parallels are equal, or rather equivalent, to one another.

Sch.—The equality of ▱s proved by the Method of Indivisibles.

Use.—The Prop. applied to convert a ▱ into a rect. of eq. area. The linear units in the base multiplied by the linear units in the altitude of a ▱ gives the Area.

36. 116. Parallelograms on eq. bases, and between the same ∥s are equal.

Use.—The Construction of the *Diagonal Scale*, and its application.

37. 118. Triangles on the same base and between the same ∥s, are equal.

Half the product of the base and altitude of a △ gives the Area.

38. 119. Triangles on eq. bases and between the same ∥s, are eq. to one another.

Sch.—By dividing the base into eq. pts., and joining the .s of division to the vertex a △ is divided into eq. parts.

Use.—This Prop. also enables us from any . in a side of a △ to divide it into two eq. parts.

39. 120. Eq. △s on the same base and on the same side of it are between the same parallels.

The *loci* of the vertices of eq. △s on the same base, form a st. line

40. 121. Eq. △s on eq. bases in the same st. line, and towards the same parts, are between the same parallels.

Sch. Cor. 1.—A ∥ to the base of a △ through the middle . of one side will bisect the other side.

2.—The lines joining the middle . s of the three sides divide the △ into four eq, △s.

Extension of the Proposition,—the vert. $\angle$ of a $\triangle$ is $<, =,$ or $>$ a rt. $\angle$ as the sq. on the base is $<, =,$ or $>$ the sum of the squares of the sides.

Remarks.

1. 143. The First Book founded entirely on the Defs. Posts. & Axioms.
2. 144. Only a few of the properties of a $\odot$ mentioned.
3. 144. A threefold Division of the Book ; 1°. from Pr. 1 to Prop. 26 the properties of $\triangle$s unfolded ; 2°. from Pr. 27 to Pr. 32, those of parallel lines ; and 3°. from Pr. 33 to Pr. 48, those of parallelograms.
4. 144. The most important Propositions are Prop. 4, 8, 26, 32, 41 and 47.

BOOK II.

The Properties of rt. $\angle$d. ▱s, or Rectangles.

145. A L may be cut *internally* or *externally*.

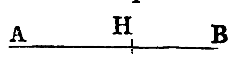
Magnitude the subject of Geometry,—Algebra and Arithmetic furnish, not proofs, but illustrations.

The Numerical Area of a Rectangle $= ab$, the altitude being represented by a, the base by b ; that of a $\triangle = \frac{1}{2}ab$.

Def. 1 & 2. p. 146. A rect is contained by any two conterminous sides; and in every ▱, any ▱ about a diam. + the complements is called the Gnomon.

Axiom. p. 146. The whole Area = the Areas of all the parts.

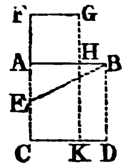
Two problems.—Prop. 11 and 14.

11. 171. To divide a L. into two parts, so that the rect. under the whole and one of the parts shall be eq. to the sq. of the other part; thus, in L  the L. is cut in the . H so that AB . HB = AH².

Or, as in 30, VI. to divide a L. in extreme and mean ratio.

The Algebraical and Arithmetical Solutions.

Cor. I. To cut a L. as AB, in extreme and mean ratio, it must first be produced in extreme and mean ratio, *i. e.* CF . FA must $= AB^2$



II.—When a L. as AB, or its equal AC, is cut in extreme and mean ratio, the rect. under the whole L. and its less segment = the sq. on the greater segments; thus, $AC . (AC—AF) = AF^2$, or $AC . HB = AH^2$

III.—Also the rect. under the whole L. as AC or AB and its greater segment = the difference between their squares, or $AC . AF$ or $AH = AC^2 — AH^2$.

Sch.—Let L. be a line cut in extreme and mean ratio, g, the greater seg. l the less, and d the difference: then 1° $L^2 + l^2 = 3\,g^2$; 2°. $(L + l)^2 = 5\,g^2$; 3°. $L . d = g . l$; and 4°. $l^2 = g . d$.

Use.—This Proposition is of frequent application, as for the construction of pentagons and the regular bodies called the Platonic Solids.

14. 179. To describe a square that shall be eq. to a given rectil figure A; *i. e.* EH^2 = rect. BD = A.

Use. 1 To find a mean propl. between two given lines; 2 to approximate to the sq. of curve-lined figures; 3. and to calculate the Areas of all plane figures.

TWELVE THEOREMS.

Prop. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10;—12, 13.

1. 146. If there be two st. lines, one A, undivided, the other BC, divided into any number of pts. as BD, ED. EC, then $A . BC = A . BD + A . DE + A . EC$.

N.B. The Alg. and Arith. illustrations are attached to each proposition.

Use. Applied to the Demonstration of the Rule for the Multiplication of numbers.

2. 148. If a st. line AB, be divided into any two pts. AD, DB then the rectangles $AB . AD, + AB . DB$ = the sq. on AB.

Numerical Multiplication may also be proved by this Prop.

3. 149. If a line, AB, be divided into any two pts. AD, DB, then the rectangle $AB . DB = AD . DB + DB^2$; or $AB . AD = AD . DB + AD^2$.

Cor. 1. $AB^2—DB^2 = (AB + DB)\,(AB—DB)$.
2.—$AD^2—DB^2 > (AD—DB)^2$ by $2\,DB . (AD—DB)$.

Also applicable to the proof of numerical Multiplication.

4. 150. **If a line AB, be divided into any two pts. AD, DB, then $AB^2 = AD^2 + DB^2 + 2\ AD \cdot DB$.**

Cor. 1. The parallelograms about the diam. of a sq. are also squares.

2.—The square of a line is four times the square of its half; thus $AB^2 = 4\left(\frac{AB}{2}\right)^2$

3.—Half the sq. of a line = 2 the sq. of half the line; thus $\frac{AB^2}{2} = 2\left(\frac{AB}{2}\right)^2$

4.—The sq. of a line will be equal to the sum of the squares of the parts + double the rect. under every distinct pair of parts.

Use 1.—In Algebra the *square of a binomial*, as $(x + y)^2 = x^2 + y^2 + 2\,xy$.

2.—This Prop. points out a practical way of extracting the Square Root of a number.

5. 154. If a line AB, A ———— C —— D —— B, be divided into two eq. parts, AC, CB, and two uneq. parts, AD, DB; then the rectangle $AD \cdot DB + CD^2 = CB^2$.

Cor. The difference of the squares of two uneq. lines, AC, CD = the rect. under their sum and diff.; *i. e.* $AC^2 - CD^2 = (AC + CD)(AC - CD)$.

Lardner's; Corollaries to this Prop. are six.

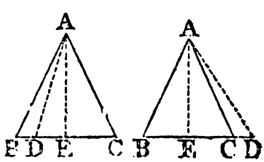
Sch.—The principal properties connected with the eq. and uneq. division of a line.

Use.—We may apply this Prop. 1°. To find *the diff. between the squares of two uneq. numbers without squaring them*; 2°. To find *quantities in Arithmetical Progression*; 3°. To establish Prop. 35, III; and 4°. To find *the value of an Adfected Quadratic Equation* in Algebra.

6. 157. **If a L. AB be bisected in C, and produced to any . D; as A ———— C ———— B .... D, then the rect.**

$AD \cdot DB + CB^2 = CD^2$.

Cor. If a line AD be drawn from the vert. A of an isosc. △ to the base or its production, the diff. between the squares of this line and the side of the △ is the rect. under the segments of the base; thus $AD^2 \sim AC^2 = BD \cdot DC$.



Use.—By aid of this Prop. the diam. of the earth may be measured.

7. 160. If a L. AB, be divided into any two parts, as AD, DB then $AB^2 + DB^2 = 2\ AB \cdot BD + AD^2$; or $AB^2 + AD^2 = 2\ AB \cdot AD + DB^2$.

Cor. 1. If AB and BD be considered as two independent lines, AD being their diff.; then $AB^2 + BD^2 = 2\ AB \cdot BD + (AB-BD)^2$

2.—Also $(AB + BD)^2$; $(AB^2 + BD^2)$; and $(AB-BD)^2$ are in Arith. Progression; the com. diff. being $2\ AB \cdot BD$.

8. 161. If a L. AB, be divided into any two parts, as AD, DB, four times the rect. under the L. and one of the pts. + the sq. of the other pt. = the sq. of the L. made up of the whole and that part; *i. e.* $4\ AB \cdot BD + AD^2 = (AB + BD)^2$

Sch. 1. Otherwise the sq. of the sum of two lines = 4 lines the rect. under them + the sq. of their diff. *i. e.* $(AD + DB)^2 = 4\ AD \cdot DB + (AD-DB)^2$

2.—Four times the sq. of half the sum = 4 times the rect. under the lines + 4 sq. of half the diff.; *i. e.* $4\left(\frac{AD + DB}{2}\right)^2 = 4\ AD \cdot DB + 4\left(\frac{AD-DB}{2}\right)^2$

Use.—The above principles are applied to Algebra and to the extraction of the sq. root.

9. 164. If a L. be divided into two eq. pts, and also into two unequal parts, as A |———C———D———| B; then the squares of the two uneq. pts. together = double of the sq. of the half line, and of the sq. of the line between the · s of section; *i. e.* $AD^2 + DB^2 = 2\ (AC^2 + CD^2)$.

Sch.—Or, 1. $AD^2+DB^2 = 2\left(\frac{AD+DB}{2}\right)^2 + 2\left(\frac{AD-DB}{2}\right)^2$

2. $AD^2+DB^2 = \frac{(AD+DB)^2}{2} + \frac{(AD-DB)^2}{2}$

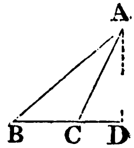
10 167. If a L. be bisected and produced to any point, as, A———C———B———D, then the sq. of the whole line thus produced + the sq. of the part produced = double of the sq. of the half line + the sq. of the line made up of the half and the pt. produced; thus $AD^2 + DB^2 = 2\ (AC^2 + CD^2)$

Sch.—Props. 9 and 10 are applicable to Algebra.

Use.—Prop. 1—10 contain the whole theory of the relations of rectangles and squares formed by lines and their parts.

1°. Given the sum and diff. of two Ms. to find them. 2°, If the Area be divided by a side the quotient = the other side. 3°. Of the five quantities depending on a rect, any two being given, the sides can be found.

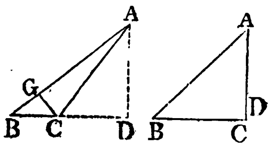
12. 173. *Important.*—In obtuse ∠d △s, if a perp. be drawn from either of the acute ∠s to the opp. side produced, the sq. of the side subtending the obt. ∠ is > the squares of the sides containing the obt. ∠ by twice the rect. under the side on which the perp. falls, and the line intercepted without the △ between the perp. and the obt. ∠.; as $AB^2 = BC^2 + AC^2 + 2\,BC\,.\,CD.$



or $AB^2 > BC^2 + AC^2$ by $2\,BC\,.\,CD$.

USE.—By this Prop. the Area of a △ may be ascertained when the three sides are known.

13. 175. In every △, the sq. of the side subtending either of the acute ∠s is < the squares containing that acute ∠ by twice the rect. under either of these sides, and the L. intercepted between the perp. let fall upon it from the opp ∠, and the acute ∠; *i. e.*, 1°. $AC^2 < (AB^2 + BC^2)$ by $2\,BC\,.\,BD$; 2°. or, by $2\,BD\,.\,BC$; and 3°. by $2\,BC\,.\,BC$.



Thus is obtained the measure of the sq. of the side subtending an acute ∠.

COR. If in the fig. to Case 2°. a perp. CG be drawn from ∠ C to AB, then the rect. $AB\,.\,GB =$ the rect. $BC\,.\,DB$

SCH.—Prop, 12 and 13 contain the Elements of Trigonometrical Analysis.

USE.—To obtain the perp, when the three sides of a △ are given; 1°. when the perp. falls *within* the base; 2°. *without* the base

The *Area* of the △ $= \dfrac{BD + DC}{2} \times AD$; or $\dfrac{BD - DC}{2} \times AD.$

REMARKS.

1. 181. Of the fourteen Propositions, the *ten* first contain the theory of the relations of the rectangles and squares on divided lines; the *twelfth* and *thirteenth* the theory of the relation between the sq. of any one side of a △, and the squares of the other two sides.

Synopsis of Book II.

Practical Results.

APPENDIX.

PART II.

Containing Books III, IV, V, & VI.

BOOK III.

PROPERTIES OF THE CIRCLE AND OF LINES IN AND ABOUT IT.

SIX PROBLEMS.

Prop. 1;—17;—25;—30;—33, 34.

Prop. 1. p. 7. To find the centre of a circle.

Cor. If in a ⊙ one L. bisects another at rt. ∠s, the cen. of the ⊙ is in the bisecting line.

Sch.—The rigour of the reasoning requires a previous proof of the conditions on which a point is *within* or *without* a circle.

17. 39. To draw a L. from a given point. either without or in the ⊙ce, which shall touch a given circle.

Sch.—From the same . two eq. tangents may be drawn.

Use.—Tangent Lines, how drawn practically;—of frequent use in Trigonometry.

25. 54. A segment of a circle being given, to describe the circle of which it is the segment.

Sch. This problem is equivalent to the following;—to inscribe a △ in a ⊙; or, to make a ⊙ pass through three given . s not in the same line.

Use.—Applied for constructing an arch,—for cutting stone, wood, metals &c., and for finding the apogee of the moon, and the eccentricity of the earth's orbit.

30. 64. To bisect a given arc of a circle.

Cor, To divide an arc into any number of eq. pts. that are powers of 2.

Sch. 1.—Except in the case of a quadrant an arc cannot be cut into 3, 5, &c., eq. pts.

2. To trisect a quadrant.

The trisection of an ∠ not effected by *Euclid's* Geometry;—what required for the trisection of an arc;—the *trisectrix* one of the trochoidal curves.

33. 72. On a given L. to describe a segment of a circle, which shall contain an ∠ = a given rectil. ∠.

Sch.—To divide a ⊙ into any number of eq, pts. of which the perimeters also are equal.

Use—Of extensive Application, as, 1. Given the distances of three landmarks to find their distancês from the place of observation. 2. To ascertain the distance from two stations; 3. Given the base and vert. ∠ to find the locus of the vertex; 4. Given the vert. ∠, the base, and the area to construct the △; 5. Through three . s to draw lines so as to make an equil. △; 6. Given the ∠ = the verr. ∠, and the base, to find the locus of the vertex. 7. Given the base, the vert. ∠ and the perp. from one end of the base to the opp. side, to construct the △.

34. 76. From a given circle to cut off a segment, which shall contain an ∠ = a given rectil. ∠.

USE 1.—By Prop. 33 and 34, if three observations be taken, the eccentricity of the earth's annual orbit, and its aphelion may be found. 2. Also in optics, to ascertain the . where two uneq. lines will appear equal.

THIRTY-ONE THEOREMS.

Prop. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. 16;—18, 19, 20, 21, 22, 23. 24;—26, 27, 28, 29;—31, 32;—35, 36, 37.

2. 9. **If any two . s be taken in the ⊙ce of a circle, the st. line which joins them shall fall within the circle.**

COR. 1. A st. line cannot cut the ⊙ce of a ⊙ in more than two points.
2. A st. line which touches a circle meets it only in one point.
3. A circle is concave towards its centre.

SCH.—Commandine's *direct* proof of Prop. 2.

USE.—A globe on a plane surface touches the plane in only a single point.

3. 11. **If a st. line through the cen. of a ⊙ bisect a st. line not through the centre, it shall cut it at rt. ∠s; and *conversely*.**

COR. 1. A L. bisecting a chord at rt. ∠s passes through the cen. of the circle.
2. All chords ∥ to the tang., at either extremity of the diam. are bisected by the diam.
3. The Line bisecting the com. chord of two ⊙s at rt. ∠s passes through the centres of both ⊙s.
4. When in a ⊙ there are several chords ∥ to each other, the *locus* of their . s of bisection is in that diam. which is at rt ∠s to them: and if the L. which bisects one chord be a perp., that line bisects all parl. chords at rt ∠s.

USE. Various, as 1. Given a ⊙ to find its centre; 2. Given an arc to find the cen. of the ⊙ of which it is an arc; 3. Through three . s not in a st. line to draw a ⊙; and 4. In Trigonometry.

4. 14. **If in a ⊙ two st. lines cut one another, which do not pass through the centre, they do not bisect each other.**

COR. No ▱ except a rectangle can be inscribed in a circle.

USE.—Applied to determine the eccentricity of the Sun's apparent path, and the centre of rotation of an eccentric wheel.

5. 15. **If two circles cut one another, they shall not have the same centre.**

SCH. Two concentric ⊙s cannot meet,—and that with the less rad. is within the other.

6. 16. **If one circle touch another internally, they shall not have the same centre.**

SCH.—Props. and 6 may be combined into one.

7\. 17. **If any . which is not the cen. be taken in the diam. of a ⊙, then, 1st, of all st. lines from it to the ⊙ce, the *greatest* is that in which the cen. is, and the other pt. of that diam. is the *least*; and 2nd, of any other st. lines, that which is nearer to the line through the centre is always *greater* than the one more remote; also 3rd, those lines which make eq. $\angle$s with the diam. are *equal*; and, 4th, from the same . there can be drawn *only two equal* st. lines, one on each side of the diam.**

SCH.—*Maximum* and *minimum* of a revolving line cut by a ⊙ce.

USE.—Of the arcs of great ⊙s from the pole to the horizon the *greatest* is that part of the meridian which passes through the zenith. The Prop. may be used for showing aphelion and perihelion distance.

8\. 20. If any . be taken without a ⊙, and st. lines be drawn from it to the ⊙ce, whereof one passes through the cen.; 1st, those which make eq. $\angle$s with the line through the cen, are *equal*; 2nd, of those which fall upon the *concave* ⊙ce, the *greatest* is that which passes through the cen.; and of the rest that nearer to the one passing through the cen. is always *greater* than one more remote; but, 3rd, of those which fall upon the *convex* ⊙ce, *the least* is that between the given . without the ⊙ and the diam., and of the rest, that nearer to the least is always *less* than one more remote; and 4th, only *two eq. lines* can be drawn from the same . to the ⊙ce, one upon each side of the line passing through the centre.

SCH. 1. The *concave* and *convex* pts of the ⊙ce determined by tangents.

2\. What is meant by the *distance* of a . from or to a line.

Of all st. lines from a . out of a line to a given line, the perp. is the *least*, and the lines nearer to the perp. are *less* than those more remote; and to a line on one side of the perp. there is only one eq. line on the other side of the perp.

USE.—If a tang. and sec. be drawn to the same ., the tang. is $<$ sec., but $>$ ext. pt. of sec.

2\. By Pr. 7 & 8, and Ax. A, III, 1st, When one ⊙ is within another, without touching, the distance of the centres, are $<$ the diff. of the radii; and *conversely*. 2nd, When two ⊙s lie each without the other and do not meet the distance of the centres is $>$ the sum of the radii; and *conversely*.

9\. 24. If a . be taken *within* a ⊙, from which there fall more than two eq. st. lines to the ⊙ce, that point is the centre of the ⊙.

Cor. 1. From any other . than the cen. only two eq. st. lines can be drawn to the ⊙ce, whether the . be within, or without the ⊙.
2. From three . s given not in the same st. line the ⊙ce of the ⊙ may be found.

Sch.—This Prop. gives the *criterion* for determining the cen. of a ⊙.
Uses. Applied, 1st, To draw a ⊙ through three given . s ; 2nd, To find the cen. of a given ⊙ ; and 3rd, To determine the centre of an arc of a ⊙.

10. 25. One ⊙ce of a ⊙ cannot cut another in more than two parts.

Sch.—If the ⊙ces of ⊙s coincide in three . s they will coincide in every point.

11. 26. If one ⊙ touch another *internally* in any ., the st. line joining their centres, being produced shall pass through the . of contact.

Sch.—When the distance between the centres = the diff. of the radii the ⊙s touch *internally*.
Use 1.—A *practically* useful method for drawing an *oval* on any given major axis may be derived from this Prop.
2. On the same principle a *Spiral* is described.
A *Spiral* defined and some of its varieties.

12. 29. If two ⊙s touch *externally* in any ., the st. line joining their centres shall pass through that . of contact.

Use.—The drawing of a *Serpentine Line*, or *cima recta*, depends on this Prop.

13. 30. One ⊙ cannot touch another in more points than one, whether it touches it on the inside or the outside.

Sch. 1, 2.—A *direct* method of demonstration substituted for the *indirect*.
Use.—Prop. 10, 11, 12 and 13, explain the motion of the Planets in Epicycles.

14. 32. Eq. st. lines in a ⊙ are equally distant from the centre; and *conversely*.

Sch.—A principle employed in Prop. 14 & 15 is this ; if $A + B = C + D$, then if $A = C$, $B = D$; if $A > C$, $B < D$; if $A < C$, $B > D$.

15. 33. The diam. is the greatest st. line in a ⊙ ; and of all others that nearer to the cen. is always greater than one more remote : and the greater is nearer to the cen. than the less.

Sch. The *longest* chord is the diam. ; the shortest through a given ., that perp. to the diam.

USE. Prop. 14 & 15 employed, 1°. To show that ⊙s of latitude diminish towards the poles ; 2°. In constructing the *Astrolabe*. 3°. In determining the position of greatest effect for a given lever.

16. 36. The st. line at rt. ∠s to the diam. of a ⊙, from the extremity of it, falls *without* the ⊙ ; and no st. line can be drawn from the extremity between that st. line and the ⊙ce, so as not to cut the ⊙ ; *or*, which is the same thing, no st. line can make so great an acute ∠ with the diam. at its extremity, or so small an ∠ with the st. line at rt. ∠s to it, as not to cut the ⊙.

COR. 1. If a st. line be drawn at rt ∠s to any diam. of a ⊙, from its extremity, it shall touch the ⊙ at the extremity ; and a st. L. touching the ⊙ at one . shall touch it at no other point.

2. By 28, I. st lines at rt. ∠s to the extremities of the same diam. are parallel.

3. Tangents to a ⊙ from the same . are equal.

SCH. The 16th Prop. may be proved *directly*.

USE. By this Prop. we can prove, 1°. The *infinite divisibility* of linear magnitudes ; 2°. The distances and heights of objects on the earth's surface, when they are situated on the verge of the natural horizon.

18. 41. If a st. L. touches a ⊙, the st. L. from the cen. to the . of contact shall be perp. to the line touching the ⊙.

SCH. Prop. 16 & 18 may be regarded as the *converse* of each other.

USE. To draw a tang. to each of two given circles.

19. 42. If a st. L. touches a ⊙, and from the . of contact a st. L. be drawn at rt. ∠s to the touching line, the cen. of the ⊙ shall be in that line.

SCH. In concentric ⊙s all chords of the greater ⊙ touching the less are eq. and bisected at the . of contact.

USE. Tangent lines are employed, 1°. in Optics to determine the part of the earth enlightened by a meteor, &c. 2°. To ascertain the earth's diam. ; 3°. To explain the Phases of the Moon ; 4°. To trace the ⊙ of the physical horizon, and 5°. in Dialling to calculate the Hour Lines.

20. 44. The ∠ at the cen. of a ⊙ is double of the ∠ at the ⊙ce on the same base, *i e.*, on the same part of the ⊙ce.

COR. Any ∠ at the ⊙ce is measured by half the arc on which it stands.

SCH. 1. The reasoning assumes, that among 4 Ms, A, B, C, D, if $A = 2B$, and $C = 2D$, then $A + C = 2(B + D)$; also that if $M = 2M'$, and $(M - a) = 2(M - a)$ then the *remainder* of $M = 2$ the remainder of M'. Another method of proof given for Case 3. Prop. 20.

2. If *Euclid's* def. of an ∠ be strictly adhered to, Prop. 20 not universally true.

3. Demonstrations may sometimes be arranged as a Simple Equation.
USE. This Prop. applicable to Trigonometry and Astronomy.

21. 47. **The $\angle$s in the same segment of a $\odot$ are eq. to one another.**

COR. If on the base of a $\triangle$, there be described a seg of a $\odot$, the vertex of the $\triangle$ shall fall *without*, or *within*, or *upon the arc*, as the vert. $\angle$ is $<$, $>$, or $=$ the $\angle$ in the segment.

SCH. The $\angle$ at the $\odot$ce has for its measure one half of the arc on which it stands.

USE. Applied, 1°. To construct a building so that the spectators may all see an object under the same $\angle$; 2°. To bisect an $\angle$; 3°. To construct a fig, representative of the distance of the place of observation from an object. 4°. To draw the arc. of any $\odot$ on a large scale.

22. 49. The opp. $\angle$s of any quadril. figure inscribed in a $\odot$ are together eq. to two rt. $\angle$s; and COR. 1. *conversely.*

COR. 2. If any side of a quadril. in a $\odot$ be produced the ext. $\angle$ = int. opp. $\angle$.
3. If two chords cut off sim. segments from the same or different $\odot$s, the other segments will also be similar.
4. If opp. $\angle$s of a quadril. be equal they must be both rt. $\angle$s.

USE. Applicable to the Construction of the Tables of Cords, and in Trigonometry.

23. 51. On the same st. L. and on the same side of it there cannot be two sim. segments of $\odot$s, not coinciding with one another.

SCH. This Prop. the same in principle as the 7th bk. I.

24. 52. Similar segments of $\odot$s on eq. st. lines are equal to one another.

COR. 1. Sim. segs. having eq chords have also eq. arcs.
2. Sim. segs. having eq. chords are parts of eq. circles.
3. If the radii, and $\angle$s of sectors are eq., the sectors themselves are equal.

USE. By this Prop. curved lined figures are often reduced to rectilineals.

26. 56. In eq. $\odot$s, eq. $\angle$s stand upon eq. arcs, whether they be at the centres or the $\odot$ces.

COR. 1. If the opp. $\angle$s be eq. their opp. diagonal must be a diam., and the seg a semicircle.
2. In the same or eq. $\odot$s one central or circumferential $\angle$ is $<$, $=$, or $>$ another, as the arc of the one is $<$, $=$, or $>$ the arc of the other.
3. The diameters intersecting at rt. $\angle$s divide the $\odot$ce into four eq. arcs.
4. When the sum of the central $\angle$s = 4 rt. $\angle$s, the sum of their arcs = the whole $\odot$ce.

5. When the sum of the $\angle$s at the $\odot$ce $=$ 2 rt. $\angle$s, the sum of their arcs also $=$ the whole $\odot$ce.

6. Sim. arcs. of eq. $\odot$s are equal.

7. Par. chords of a $\odot$ intercept eq. arcs.

8. If two chords intersect *within* a $\odot$, the sum of the intercepted arcs $=$ the arc which the $\angle$ would intercept at the $\odot$ce, that is eq. to the $\angle$ under the chords.

9. If two chords intersect at a . *without* a $\odot$, the difference of the arcs which they intercept is $=$ to the arc, which an $\angle$ would intercept at the $\odot$ce that is eq. to the $\angle$ under the chords.

USE. Applied to find the true central $\angle$ of an imperfect theodolite.

27. 59. In eq. $\odot$s the $\angle$s which stand upon eq. arcs are eq. to one another, whether they be at the centres or the $\odot$ces.

COR. 1. In the same or eq. $\odot$s, the sectors on eq. arcs are equal ; and *conversely*.

2. If the chords of a $\odot$ are parallel, they intercept eq. arcs ; and *vice versâ*.

SCH. 1. What is true of eq, $\odot$s, is true of eq. arcs in the same $\odot$.

2. The sum of the $\angle$s at the cen. of a $\odot$ $=$ 4 rt $\angle$s ; and the sum of the $\angle$s at the circumference $=$ 2 rt $\angle$s.

USE. By this Prop. a . parallel through a given . may be drawn.

2. To ascertain the Area of a Sector.

28. 61. In eq. $\odot$s, eq. st. lines cut off eq. arcs, the greater $=$ the greater, and the less to the less.

29. 62. In eq. $\odot$s, eq. arcs are subtended by eq. st. lines.

COR. 1. In the same or eq. $\odot$s, eq. sectors stand on eq. arcs; and *conversely*.

2. St. lines which intercept eq. arcs are parallel; and par. st. lines intercept eq. arcs.

USE. In Spherical Trigonometry Prop. 26, 27, 28 and 29, are of continual use.

31. 66. In a $\odot$, the $\angle$ in a semicircle is a rt. $\angle$, but the $\angle$ in a seg. $>$ a semicircle is $<$ a rt. $\angle$; and the $\angle$ in a seg. $<$ a semicircle is $>$ a rt. $\angle$.

COR. If one $\angle$ of a $\triangle$ be eq. to the other two, it is a rt $\angle$.

SCH. 1. *Conversely*, The seg. containing an acute $\angle$ is $>$ a semicircle ; and the seg. containing an obtuse $\angle$ is $<$ a semicircle.

2. *Lardner's* elegant and brief demonstration of 31. III.

68. · USE. Problems derived ; 1°. From a . *in* a line or at its extremity to draw a perp. ; 2°. From a . *without* a line to draw a perp. ; 3°. From a . without a $\odot$ to draw a tangent ; 4°. By means of a Square to find the centre of a $\odot$; 5°- To try if Squares are exact.

32\. 69. If a st. L touches a $\odot$, and from the . of contact a st L. be drawn cutting the $\odot$; the $\angle$s which this line makes with the line touching the $\odot$, shall be eq. to the $\angle$s in the alternate segs of the $\odot$.

Cor. 1. *Conversely,* If from the end of a line cutting the $\odot$, &c.

2\. If two or more $\odot$s touch each other, and through the . of contact two st. lines be drawn meeting their $\odot$ces, the chords of the intercepted arcs will be parallel.

3\. If two or more $\odot$s touch each other, at a com. point of contact, any line passing through the . of contact will cut off sim. segs. from each.

4\. In an equil. $\triangle$, if the sides be bisected, and st. lines be drawn joining the . s of bisection, of those lines, two will be tangents to the $\odot$, which passes through the ends of the other line, and the ang. point opp. to that line.

5\. Tangents through the extremities of the same chord, make the $\angle$s on the same side equal.

6\. If tangents are par., the line joining the . s of contact is a diam.

Use. 1. This Prop. preliminary to the 33rd, and required for various Theorems.

35\. 77. *Very important.* If two st. lines cut one another within a $\odot$, the rect. under the segments of one of them $=$ the rect. under the segments of the other.

Cor. *Conversely,* If the rectangles be equal &c.

Sch. Or, If two chords of a $\odot$ cut one another, the rectangles under their segs. terminating in the . of section shall be eq.

The terms *ordinate* and *abscissa* explained.

Use. 1. If of two eq. $\odot$s, the centres be each on the $\odot$ce of the other, and a com. chord be drawn $\parallel$ to the line joining the centres, then the lines joining the . s where the com. chord cuts the $\odot$s and the extremities of the lines joining the centres, form $\square$s; 2. To find a line which is a fourth propl. to three given lines, or a third propl. to two given lines.

36\. 81. *Important.* If from any . *without a* $\odot$ two st. lines be drawn, one of which cuts the $\odot$, and the other touches it; the rect. under the whole line which cuts the $\odot$, and the part of it without the $\odot$, shall be $=$ the sq. of the L. which touches it.

Cor. 1. If from any . *without* a $\odot$ there be drawn two st. lines cutting it; then the rectangles under the whole lines and the parts of them without the $\odot$ shall be $=$ to one another.

Cor. 2. If from the same . two tangents be drawn to the same $\odot$, they are equal.

Sch, Or, If any chord of a $\odot$ be prod. to cut a tang. to the same $\odot$, the sq. of the tang. shall be $=$ the rect. under the segs. of the chord produced.

USE 1. In any semicircle if from the extremities of the diam. chords be drawn intersecting within the semicircle; then the sum of the rectangles under any two such intersecting chords and the sections of the chords between the extremities of the diam. and the intersecting ., shall = the sq. on the diam.

USE. 2. The *Art of taking a true Level* is deduced from this proposition; 1° Plan of a Field Book; 2° The measuring of an Ascent. 3° Correction for curvature. 4° Deviation of the horizontal from the true level. 5° Distances of the horizontal boundary. 6° Sum of the horizons when taken. 7° The earth's diam. ascertained.

37. 85. If from a . *without* a ⊙ there be drawn two lines, one of which cuts the ⊙, and the other meets it; if the rect. contained by the whole L. which cuts the ⊙, and the part of it without the ⊙, be eq. to the sq. of the line which meets it, the L. which meets shall touch the ⊙.

COR. Tangents from the same . without a ⊙ are equal.

USE 1. Through two given . s to describe a ⊙ touching a given ⊙.

2. Prop. 35, 36 and 37 are amongst the most important in Plane Geometry. The earth's diam. calculated from them by *Maurolico* in the 16th century.

REMARKS.

1. 87. The Propositions of Book III, classified under *five* general heads.
2. 87. Of the 37 Propositions six only are Problems.
3. 88. Twelve others have been deduced.
4. 88. Problems might be given for drawing ⊙s which are tangents to two or three given ⊙s, &c.
5. 89. Several of the Principles of Levelling and Surveying, and of making Geographical and Astronomical Observations laid down.

BOOK IV.

METHODS OF CONSTRUCTING REGULAR STRAIGHT-LINED FIGURES IN AND ABOUT A CIRCLE &C.

91. Excepting Pr. A, Theor., the fourth book consists entirely of Problems.

This book of essential service in Astronomy, and in Civil and Military Engineering.

Def. 1—7 pp. 92, 93. Euclid's Definitions.
8—10 97. Definitions additional to those of *Euclid*.

USE. If one ⊙ be inscribed in an equil. △ and another circumscribed about it, the ⊙s are *concentric*, and the rad. or the diam of one is *double* the rad. or the diam. of the other.

A. 105. THEOR. A circle may be described about any reg. polygon, or inscribed within it; and *conversely*.

USE. 1. The construction of a reg. polygon; 2. To inscribe a polygon in a given ⊙; 3. To circumscribe a polygon; 4. The Area of a reg. polygon; 5. The Area of a ⊙.

6. 106. PROB. To inscribe a square in a given circle.

COR. The Sq. on the rad. of an inscribed ⊙ is $\frac{1}{2}$ the sq. inscribed in a ⊙, and $\frac{1}{4}$ the sq. on its diameter.

USE. By bisecting the arcs, and joining the .s of bisection an *octagon* may be formed, &c.

7. 107. To circumscribe a square about a given circle.

COR. In the same ⊙ the circumscribed square = twice the inscribed square.

USE 1. To inscribe and to circumscribe a reg. *octagon*.

2. A reg octagon inscribed in a ⊙ = rect. under the sides of the inscribed and circumscribing squares. 3. If a quadril. be circumscribed about a ⊙, any two of its opp. sides = $\frac{1}{2}$ the perimeter.

8. 109. To inscribe a circle in a given square.

SCH. To inscribe a circle in a given quadrant.

9. 111. To circumscribe a circle about a given square.

10. 112. To construct an isosc. △, having each of the ∠s at the base double of the third, or vertical angle.

USE 1. In the fig. constructed for Prob. 10, the side AC inscribed in the smaller ⊙ACD = the side of a *pentagon* in that ⊙, and also = the side of a reg. *decagon* in the larger ⊙ BDE.

2. On the side DC being produced to meet the ⊙ BDE in F, and FB being joined, the ∠ ABF = *three* times ∠BFD.

3. To quinquisect, *i. e.*, divide a quadrant into five equal parts, &c.

11. 115. To inscribe an equil. and equiangular pentagon in a given circle.

SCH. 1—4. Remarks on reg. polygons. 5. Formulæ for determining the relative magnitudes of the ∠s of isosc △s to be used in the construction of reg. polygons.

USE. 1. To draw a △ = in area to a given polygon.

2. The lines joining the alternate ∠s of a reg. pentagon, will form another reg. pentagon; and the .s of intersection of the alternate sides produced will also form another reg. pentagon.

12. 118. To circumscribe an equil. and equiang. pentagon about a given circle.

SCH. If the ⊙ce of a ⊙ be divided into any number of parts, the chord joining the . s of division shall include a reg. polygon, inscribed in the ⊙; and the tangents through those . s shall include a reg. polygon of the same number of sides circumscribed about the ⊙.

13. 120. To inscribe a circle in a given equil. and equian. pentagon.

14. 121. To circumscribe a circle about a given equil. and equiang. pentagon.

SCH. 1. To circumscribe a reg. polygon about a given circle.
2. To inscribe a circle in a reg. polygon.
3. To circumscribe a circle about a given polygon.

ADDENDA to 14, IV. 1. To analyze the conditions on which the drawing of a reg. *decagon* and a reg. *pentagon* depends.
2. To demonstrate, that the sq. on the side of a reg. pentagon inscribed in a ⊙ = the sum of the squares of the rad., and of the side of the inscribed decagon.

15. 124. To inscribe an equil. and equiang. hexagon in a given circle.

COR. 1. The side of a reg. hexagon inscribed in a ⊙ is = the rad., or semidiam, of the circle; or the chord of 60° = the rad.
2. An equil. △ would be inscribed by joining the alternate . s in the hexagon.
3. Every equil. fig. inscribed in a ⊙ is equiangular.

SCH. 1. The opp. sides of a hexagon are parallel.

USE 1. On a given L. to describe a reg. hexagon.
2. The inscribed hexagon in a ⊙ is three-fourths of the area of the circumscribed hexagon.
4. Half the rad. = sine of 30°.
5. The inscribed hex. and the successive bisections of its arcs, the ground-work for finding the *approximate ratio of the* ⊙; *i. e., of a* ⊙ *to its diameter. O* r a Square may be taken as the ground-work of the process.

16. 128. To inscribe an equil and equiang. quindecagon in a given circle.

COR. The only reg. st. lined figures which can be placed, side by side, so as to make a continuous plane surface, are the equil. △, the square, and the hexagon.

SCH. 1. To circumscribe a reg. quindecagon about a circle.
2. To find the arc subtending a side of a *reg. thirty-sided* figure.

USE. This Prop. opens the way for the *construction of other polygons.*

OBSERVATIONS ON POLYGONS.

1. 130. I. The *four* known ways of dividing a ⊙ geometrically.

2. 130. II. To many polygons we must apply an *approximate process.*

1. In a given ⊙ to inscribe any reg. rt. lined fig.; or to divide the ⊙ce of a ⊙ into any assigned number of eq. parts.
2. Approximative Method,— for the Heptagon, Nonagon, &c.

3. 132. III. 1st. To find the magnitude of an ∠ at the centre of a reg. polygon.
2nd. To find the magnitude of an ∠ formed by two adjacent sides of a polygon.

4. 133. IV. On a given rt. Line AB to construct a reg. polygon.
1. The Formula, $\pi - \frac{2\pi}{n}$ = one of the eq. ∠ s of the polygon.
2. The Line × tabular rad. = units of length in the rad. of the circumscribing ⊙.

5. 134. V. The Area of a reg. polygon $= \frac{n\,AB}{2} \times$ the perp. from the centre.
By using the Table, $AB^2 \times$ tabular Area = Area of reg. polygon.

6. 135. VI. Dodson's Tables for calculating and constructing Polygons of not more than 12 sides.
1°. When the length of the side = 1.
2°. When the radius of circumscribed ⊙ = 1.
3°. When the radius of inscribed ⊙ = 1.
4°. When the area = 1.

7. 137. VII. Polygonal numbers, their nature and construction.
To find the numbers which bear the name of an n*—sided figure.*

8. 138. VIII. The nature and Construction of *Star-shaped Polygons, i. e.* reg. polygons with *re-entrant* angles.

Remarks.

1. 139. Classification of the 16 Problems under *five* general heads.
2. 139. Theoretical Reasoning in Geometry guides to most important Practical results.

BOOK V.

The Theory of Proportion, or of the Comparative Magnitudes of Plane Figures.

141. Book V. independent of the preceding books,—its subjects, *ratio* and *proportion*.
Illustrations to be derived from Algebra and Arithmetic.

The *Criterion* for determining the equality of Ratios.
Applicable to all *magnitudes*, incommensurables and commensurables.
Ratios when they are the same.

6\. 152. Magnitudes which have the same ratio are called *proportionals*.

7\. 153. When a Magnitude has a *greater* ratio,—when a *less*.

8\. 153. Analogy or Proportion is the similitude of ratios.

9\. 153. Proportion consists in three terms at least.

10\. 153. The *Duplicate Ratio*,—when three magnitudes are proportionals.

The Duplicate Ratio expressed algebraically and arithmetically.
Such magnitudes in *continued* proportion. A mean Proportional.
Double Ratio and *Duplicate* Ratio not to be confounded.

11\. 154. Triplicate Ratio, quadruplicate, &c.

A 155. *Of Compound Ratio*,—with examples.

B. 155. Progression on the lengths of chords producing musical sounds.

The 1st : 3rd = 1st ∾ 2nd : 2nd ∾ 3rd.
Three st. lines when in *Harmonical* Progression. Harmonical Mean.

12\. 156. Terms when *homologous*, or corresponding.

TECHNICAL WORDS TO DENOTE CHANGES IN THE ORDER OF PROPORTIONALS.

1°.—*For Four Proportionals.*

156—8. Def. 13. *Permutando*, or *alternando* ; Def. 14. *Invertendo* ; Def. 15. *Componendo* ; Def. 16. *Dividendo* ; Déf. 17. *Convertendo*. Conjungendo.

2°.—*For any number of Proportionals above Two.*

158—160. Def, 18. *Ex æquali* (sc. *distantia*) or *ex æquo* ; Def. 19. *Ex æquali*, or *ex æquo ordinate* ; Def. 20. *Ex æquali in proportione perturbatâ seu inordinatâ*, or *ex æquo pertubate*.

POSTULATES 1 and 2 p. 161.

3\. 173. Three Ms being given, A,B,C, there is a 4th M., as x, to which C has the same ratio, as A to B, *i. e.*, $A : B = C : x$.

AXIOMS.—1—4 p. 161.

Algebraical Expressions, &c., p. 161.

ONE PROBLEM.—PROP. N.

N. 237. To find a common measure of two lines.

COR. 1. The greatest com. meas. of the rem. and lesser M. is also the greatest com. measure of the two Ms.

2. Any aliquot part or subm. of a com. meas. is a com. measure.

3. By repeating the process with the rem. and the lesser M., and again with the new rem. (if there be one) and the preceding, and so on, *the greatest com.* meas. of two given commensurable Ms. may be found.

4. Any two commensurable Lines are to one another as the numbers denoting the no. of times that they respectively contain their com. measure.

239. SCH. When Ms are incommensurable.

USE. To find the greatest com. measure of two numbers.

EUCLID'S THEOREMS ARE TWENTY-FIVE,—THE SUBSIDIARY FOURTEEN.

Prop. 1, 2, 3, 4, 5, 6 ;—A, B, C, D ;—7 A ;—7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 ;—E ;—20, 21, 22, 23, 24, 25 ;—F, G, H, K, L, M ;—O, P.

1. 162. If any no. of Ms be equims. of as many, each of each; what *m* soever any one of them is of its part, the same shall all the first Ms be of all the other.

COR. The sum of the equimultiples = the equimultiple of the sum.

SCH. If to a *m* of a *M.* by any number a *m* of the same M. by any number be added, the sum will be the same *m* of that *M* that the sum of the two numbers is of unity.

COR. 1, Thus, as $A = m\ E$, $B = n\ E$, $C = p\ E$, &c., $A + B + C = (m + n + p)\ E$.

2. Also $m\ E + n\ E + p\ E = (m + n + p)\ E$.

2 164. If the first M be the same *m* of the second that the third is of the fourth, and the fifth M the same *m* of the second that the sixth is of the fourth; then shall the first + the fifth be the same *m* of the second, that the third + the sixth is of the fourth.

COR. If AB, BG, GH, &c., be mults. of C. and as many, DE, EK, KL, &c., the same *ms* of F, each of each, then AH, *i. e.*, (AB + BG + GH &c.) the same *ms* of C, that DL, *i. e.*, (DE + EK + KL) is of F.

SCH. If the 1st of three Ms contain the 2nd as often as there are units in a certain number;—and if the 2nd contain the 3rd also as often as there are units in a certain number, the 1st will contain the third as often as there are units in the product of these numbers.

3. 165. If the 1st be the same m of the 2nd, which the 3rd is of the 4th; and if of the 1st and 3rd there be taken equims; these shall be equims, the one of the 2nd, and the other of the 4th.

COR. If A, A′ be equims of B, B′, and also of C, C′; and if B be a m of C, then B′ is the same m of C′.

SCH. If $A : B = C : D$, and m A, m C, n B, n D be taken, then $m\,A : n\,B = m\,C : n\,D$.

COR. When $n = 1$, then $m\,A : B = m\,C : D$.

4. 167. If the 1st of four Ms has the same ratio to the 2nd which the 3rd has to the 4th; then any equal *ms* whatever of the 1st and 3rd shall have the same ratio to any equims of the 2nd and 4th.

COR. 1. If the 1st has the same ratio to the 2nd, which the 3rd has to the 4th, then also any equims whatever of the 1st and 3rd shall have the same ratio to the 2nd and 4th; *so*, the 1st and 3rd shall have the same ratio to any equims. whatever of the 2nd and 4th.

2. If 4 Ms are propls, they will be propls by *inversion*.

3. If $A : B = C : D$, then $\frac{A}{2} : \frac{C}{2} = \frac{B}{3} : \frac{D}{3}$.

APPL. Hence in the rule for Simple Proportion in Arith.,—divide the 1st and 2nd terms by any com. measure and make use of the results.

5. 170. If one M. be the same m of another, which a M. taken from the 1st is of a M. taken from the other; the rem. is the same m of the rem., that the whole is of the whole.

SCH. If from a m of a M. by any number, a m of the same M. by a less number be taken away, the *rem.* will be the same m of that M. that the diff. of the numbers is of unity.

COR. When $m - n = 1$, then $m\,A = n\,A =$; or $2\,A - A = A$.

6. 171. If two Ms be equims of two others, and if equims of these be taken from the first two; the rems are either = these others, or equims of them.

SCH. Pr. 1—6 chiefly useful to establish, by the method of equims. the Props. which follow.

Postulate. Three Ms, A, B, C, being given, there is a 4th M., as x, to which C has the same ratio as A to B; *i. e.*, $A : B = C : x$.

COR. 1. If A : B = C : D, but C : D > or < E : F, then A : B > or < E : F.

2. If A : B > or < C : D, and C : D = E : F, then A : B > or < E : F.

12. 189. If any number of Ms be propls, as one of the antecs is to its conseq., so shall all the anteceds taken together be to all the conseqs.

13. 191. If the 1st : 2nd = 3rd : 4th, but the 3rd : 4th > 5th : 6th; the 1st : 2nd > 5th : 6th.

COR. If A · B > or < B : E, but C : D = E : F, then A : B > or < E : F.

14. 193. If the 1st : 2nd = 3rd : 4th; then if the 1st > 3rd, the 2nd > 4th; and if =, =, and if <, <.

COR. If A : B = C : D, and if B > = or < D, then A > = or < C.

15. 194. Ms have the same ratio to one another which their equims have.

COR. 1. Ms have the same ratio to one another which their eq. submults. have.

2. If A : B = C : D, then m A : m B = n C : n D.

3. Also $\frac{A}{2} : \frac{B}{2} = \frac{C}{3} : \frac{D}{3}$.

16. 196. *Alternando.* If 4 Ms of *the same kind* be propls, they shall also be propls when taken alternately.

197. USE 1. If to the terms of a rat. the same M. be added, the rat. will be unchanged, increased, or diminished, according as it is a rat. of equality, of less inequality, or of greater inequality.

2. If all the terms, or any two homol. terms, or the terms of either of the ratios of proportion, be multiplied or divided by the same number, the resulting Ms will remain propl.

17. 199. *Dividendo.* If Ms taken jointly, be propls, they shall also be propls when taken separately; *i. e.*, if two Ms. together have to one another the same ratio which two others have to one of these, the rem. one of the first two shall have to the other the same ratio which the rem. one of the last has to the other of these.

COR. 1. *Convertendo.* If A : B = C : D, then A : A $\sim$ B = C : C $\sim$ D.

2. The greatest of four propls + the least > the other two.

3. If A : B : C, then A + C > 2 B; and $\frac{A + C}{2}$ > B.

Use. The arith. mean between 2 Ms is > the geom. mean.

Cor. Rs. compd. of any no. of eq. Rs. in the same order are equal to one another.

Sch. Varied thus ;—If the 1st M be to to the 2nd, *as* the 3rd to the 4th ; and if the second M be to the 5th, *as* the 4th to the 6th ; then the 1st shall be to the 5th, as the 3rd to the 6th.

Use. Proportionals remain propl, *miscendo*, by mixing; *i. e.*, by using their sum and difference.

23. 219. *Ex æquo perturbato.* If there be any no. of Ms, and as many others, which taken two and two in a cross order, have the same R.; the first shall have to the last of the first Ms the same R. which the first has to the last of the others.

Sch. Different ways of announcing Prop. 23. 1. Rs compd of any no. of eq. Rs, but in reverse order, are eq. to one another; 2. Also for two or more series of Ms.

24. 222. If the first has to the second the same R. which the third has to the fourth, and the fifth to the second the same R. which the sixth has to the fourth; the first and fifth together shall have to the second, the same R. which the third and sixth together have to the fourth.

Cor. 1. The excess of the 1st above the 5th, shall be to the 2nd, *as* the excess of the 3rd above the 6th is to the 4th.

2. In any no. of Proportions,—if the 2nd is the same throughout and also the 4th ; then the sum of all the first terms is to the com. 2nd term, *as* the sum of all the third terms *is* to the com. 4th term.

Sch. Or, "If two series of Propls. have the same conseqs., the sum of the first antecs : com. conseq. = sum of the second antecs : their com. conseq."

Cor. 1. If two proportions have the same consequents, as $A : B = C : D$, and $E : B = F : D$, then $A - E : B = C - F : D$.

2. If four Ms form a proportion, $A : B = C : D$, *miscendo*, $A + B : A - B = C + D : C - D$.

3. In two series A, B, C, D, E, F, &c., and G, H, I, K, L M, &c.,—if the ratios, $A : B$, or $G : H$; $B : C$, or, $H : I$, &c., be the same in the two series, then any two combinations of the first series shall be to one another as any two similar combinations of the second series.

25. 226. If four Ms of the same kind are propls, the greatest + the least are > the other two together.

Cor. If three Ms be propls, the sum of the extremes > twice the mean, and half the sum > the mean.

The *Arith.* mean is > the *geom.* mean.

F. 227. Ratios compounded of the same ratios are the same to one another.

USE. This Prop. is extensively applied, and is one of the first steps to the higher Geometry.

O. 240. The diagonal and side of a Square are incommensurable.

P. 241. If four st. Ls., A, B, C, D, be propls, (whether commensurable or incommensurable,) the rect. under the extremes, $A \cdot D$, will be = the rect. under the means, $B \cdot C$.

USE The Theory of Proportion in Arith. and Alg. is founded on this truth.

243. *Examples of Reasoning by Proportion* 1°. —11°.

244 Remarks on Book V. by BILLINGSLEY.

BOOK VI.

THE THEORY OF PROPORTION APPLIED FOR COMPARING THE SIDES AND AREAS OF PLANE RECTILINEAL FIGURES.

247. Billingsley's Summary.
248. Identity of Form, not of size, the basis of the Comparison. The extension given to other truths.
249. What the sixth Book establishes stated in general terms.

DEFINITIONS.

1. 249. Similar rectil. figures,—conditions to be fulfilled.
2. 250. Reciprocal figures,—in what way the sides propl.
3. 250. A st. L. cut in extreme and mean ratio.
   A L. divided medially, or in *medial ratio*.
4. 251. The altitude of a figure.

SUBSIDIARY DEF. A — F.

A. 251. A L. *harmonically* divided.
B. 251. A fig. given in species.
C. 251. A fig. given in magnitude.
D—F. 252,3. A ▱ applied to a st. L.

2. *Problems* by aid of the compasses and sector; 1°. To find a fourth propl. to three given lines.

2°. To find a chord to a rad., from a Sector the rad. of the chord of which equals 4 inches.

3°. To divide a given L. into two pts, x and y, which shall be to each other as two lines or numbers.

3. The *various* Lines on the Sector, and on GUNTER's Scale.
4. Examples of the use of the Sector.
5. Other applications, and caution.

13. 302. To find a mean proportional between two given st. lines.

SCH. Other constructions;—formula for numerical calculations.

USE. 1. Any rectangular parallelogram may be reduced to an equivalent square.

2. Of three lines in cont. proportion, any two being given, to find the unknown.
3. Given one of three terms and the sum of the other two, to find the two unknown terms.
4. Of three lines in cont. proportion, if one be given and the diff. of the other two, those other two may be found.
5. To find two st. lines to contain a rect. = a given rect. and to have a given ratio one to the other.
6. To find any number of means represented by a power of 2, *minus* 1.

ADDENDA. I. To obtain two mean propls. between two given st. lines;

1°, PLATO's method of a perp. moveable along a side of a sq.

2°. PHILO's method of a graduated ruler revolving round the vertex of a rt. ∠.

3°. The method of DES CARTES, with a collection of rulers.

308. II. The Trisection of a rectilineal ∠. 1°. The Trammel of NICOMEDES, a T square with a moveable ruler.

2°. By this instrument to trisect a given ∠.

3°. COOLEY's Tentative Method.

18. 324. Upon a given st. L. to describe a rectilineal figure, similar and similarly situated to a given rectilineal figure.

SCH. 1. A more simple way for making a rectil. fig. sim. to a given rectil. fig.

2. To construct a rectil. fig. sim. to a given rectil. fig., and having its perimeter = a given st. L.

3. Figures of the same species with different areas on the same rt. L.

4. Sim. △s and polygons are to one another as the squares of their homol. sides.

USE. Nearly all the *practical methods of taking a Plan or Map*, founded on this Prop.

25. 344. *Of extensive use.* To describe a rectil. figure which shall be sim. to one and eq. to another given rectil. fig.; *i. e.* eq. in area.

Use. This Prop. employed in Euclid's 13th Bk. on the Platonic Solids.
Prob. 1°. On a given L. to construct a rt. ∠d △, the sides of which shall be in cont. or geom. progression.
Prob. 2°. The altitude of a rt. ∠d △ being given, of which the sides are in a given R., to find the sides.
Prob. 3°. Given, *a*, *b*, two sides of a △, and the diam. *d*, of the circumscribing ⊙, to find the other side, *c*.

EUCLID'S THEOREMS ARE TWENTY-THREE ; THE SUBSIDIARY TWELVE.

Prop. 1, 2, 3 ;—A, 4, 5, 6, 7, 8 ;—14, 15, 16, 17 ;—19, 20, 21, 22 ;—Lemma. 23, 24, 25, 26, 27 ;—31, 32, 33 ;—B, C. D ;—E, F, G, H, K, L, M.

1. 252. Triangles and parallelograms of the same altitude are to one another as their bases.

Cor. 1. Triangles & ▱s with eq. altitudes are one to another as their bases ; & *conversely*.
2. Any two △s or ▱s, are to one another in the R. compounded of the Rs of their altitudes and of their bases.
3. The rect. under two lines is a mean propl. between their squares.
4. If two △s or two ▱s be as their bases, they have eq. altitudes ; and if they have eq. altitudes they have eq. bases.

Sch. 1. This Proportion might have been directly inferred.
2. Propositions distinguished by the name *Variant*.
3. One quantity does not vary *as* another, because it varies *with* it.

Use. From a trapezium to cut off a third part.

2. 257. If a st. L. be drawn ∥ to one of the sides of a △, it shall cut the other sides, or these produced, proportionally; and *conversely*.

Cor. If the sides of an ∠ be cut by any no. of parallels, any two pts. of the one will have the same R. to one another, as the corresponding parts of the other.

Sch. 1. The Enunciation not sufficiently explicit.
2. The theory of Transversal lines is connected with this Prop.
3. The ways in which a st. L. may be cut in a given Ratio.
4. Parallel lines cut diverging lines proportionally.

Use 1. For the measurement of the height of an inaccessible object which casts an accessible shadow.
2. To divide a given L. into parts propl. to those of another L.

3. 260. If the ∠ of a △ be divided into two eq. ∠s by a st. L. which also cuts the base; the segs. of the base shall have the same R. which the other sides of the △ have to one another; and *conversely*.

A. 262. If the outward $\angle$ of a $\triangle$, made by producing one of its sides, be divided into two eq. $\angle$s, by a st. L., which also cuts the base produced; the segs. between the dividing line and the extremities of the base, have the same R. which the other sides of the $\triangle$ have to one another; and *conversely*.

Cor. 1. The segs. of the base produced made by the external bisector are propl. to the segs. of the base made by the internal bisector.
2. The two lines bisecting the vert. $\angle$ and its adj. ext. $\angle$, cut the base produced *harmonically*.
3. Also the two sides of a $\triangle$, and the lines which bisect the vert. & ext. vert $\angle$s are *harmonicals*.
4. If BG, BC & BD, in the same st. L. be in harmonical progression, DC, DG & DB, will also be in harmonical progression.

Sch. Case in which there is no point of external bisection &c.

Use 1. The harmonic mean obtained from the harmonic proportion.
2. Applied to Optics and Acoustics.

4. 265. The sides about the eq. $\angle$s of eq. ang. $\triangle$s are proportionals; and those which are opp. to the eq. $\angle$s are homol. sides.

Cor. 1. If diverging Ls, cut par. Ls, the par. Ls will be cut proportionally.
2. If two par. st. Ls be cut by any number of diverging Ls, the parallels will be similarly cut in the .s of section.
3. In a $\triangle$, a L from the vertex, bisecting the base also bisects the parallel to the base.
4. A par. to the base of a $\triangle$ cuts off a sim. $\triangle$.

Sch. 1. Homologous sides,—homologous terms.
2. Only in $\triangle$s, if their $\angle$s are eq. their sides about the eq. $\angle$s are propl.

5. 268. *Important.* Conversely,—If the sides of two $\triangle$s, about each of their $\angle$s, be propls, the $\triangle$s shall be eq. ang.; and the eq. $\angle$s shall be those which are opp. to the homol. sides.

269. Sch. Pr 47, 47, I, and 4, 5, VI contain the principles of every kind of rectilineal Measurement.

Use 1. The Theory of Representative Value,—the Practice of Triangulation, and various other methods of Practical Geom. depend on Pr. 4 & 5, bk. VI.
2. For Practical Purposes Eight Problems and Formulas deduced;
1°. Given the observed length of the shadows of two perp. objects, and the alt. of one, to find *the alt.* of the other.

2°. By means of a mirror placed horizontally, the ∠ of incidence being = the ∠ of reflection, to ascertain the *height* of a perp. object.
3°. To find the *height* of a perp. object, by means of two uneq. rods or poles, placed perpendicularly on a horizontal line.
4°. By means of a pole placed perp., to ascertain the *alt.* of a perp. object.
5°. By a Geom. Square to measure the *height* of an object.
6°. By the same means, with an index to point to the extremities of two objects, of which one is in the vertex of a rt. ∠; to measure the *distance*.
7° To find, by aid of the cross staff, or theodolite, the *distance* between two objects.
8°. By means of a L. of which the length is known, to find the *length* of its parallel, one end of which only can be approached.

The Proportional Compasses, Pentagraph, and Eidograph.

6. 275. If two △s have one ∠ of the one = one ∠ of the other, and the sides about the eq. ∠s propl., the △s. shall be eq. ang., and shall have those ∠s. eq. which are opp. to the homol. sides.

Cor. 1. The sides also about each pair of eq. ∠s shall be propl.
2. If through any . s of a rt. L. parallels be drawn, proportional to distances from any . A in that st. L., then their extremities will be on the rt. L. passing through A.

Use 1. Conditions of similarity between one rectil. figure and another.
2. Sim. rectil. figures may be divided into the same no. of sim. △s.

7. 278. If two △s, have one ∠ of the one = one ∠ of the other, and the sides about two other ∠s propls.; then, if each of the rem. ∠s be either < or ≮ a rt. ∠, or if one of them be a rt. ∠, the △s shall be eq. ang. and shall have those ∠s. eq. about which the sides are proportionals.

Sch. Angles when of the *same affection*.

Use. The *criteria* of the similarity of two △s; 1°. Equality of the three ∠s. 4, VI. 2°. Identity of the Rs of the respective sides, 5, VI. 3°. Equality of two ∠s, one in each △, and the identity of the Rs of the containing sides, 6, VI; 4°. Identity of the Rs of two sides in each △; the equality of an ∠ in each opp. one pair of homol. sides; and each of the rem. ∠s opp. the other pair of homol. sides < a rt. ∠, or one of them a rt. ∠.

8. 281. *Important.*—In a rt. ∠d △, if a perp. be drawn from the rt. ∠ to the base; the △s on each side of it are similar to the whole △ and to one another.

4. The doctrine of *Limits* needed for incommensurable Ms.

5. THEOR. 1°. If there be two fixed Ms, A & B, which are the limits of two others, P and Q, and if P be to Q always in the same given R. of C to D; then A shall be to B in the same R.

THEOR. 2°. If four st. Ls. be propls, (whether commensurable or otherwise) the rect. under the extremes will be = rect. under the means.

USE. The Theory of Limits applied; 1°. Rectangles having the same alt. are to each other as their bases; 2°. Rectangles are to each other as the product of their bases by their altitudes; 3°. Among other theorems the rect. contained by any two sides of a △ is eq. to the rect. contained by its alt., or perp. to the third side from the opp. ∠ and by the diam. of the circumscribing ⊙.

17. 321. COR. to Pr. 16. If three st. Ls. be propls, the rect. contd. by the extremes = the Sq. of the mean; and *conversely.*

USE. 1. The Demonstration of Proportion in Arithmetic, or *the Rule of* Three.

2. A clear dem. of 47, I. ; In a rt. ∠d △, $Hyp.^2 = side^2 + side^2$.

3. Deductions stated in other words.

4. Instances in which Props. 16 and 17 shorten the demonstrations.

1°. If from a . there be drawn two st. Ls, one a tang. to a ⊙, the other a secant, then the tang. will be a mean propl. between the whole secant and its ext seg.; 2°. If from a . there be drawn several st. Ls., cutting the ⊙, then the whole secants will be one to another *inversely* as their ext. segment.

19, 328. *Very important.* Sim. triangles are to one another in the duplicate R. of their homol. sides; *i. e.*, as the squares of their like sides.

COR. If three st. Ls. are propls., *as* the 1st *to* the 3rd, *so* is any △ on the 1st *to* a sim. and sim described △ on the 2nd.

SCH. The duplicate R. of two st. Ls. is the same with the R. of their squares.

USE. *A general method of reasoning* thus supplied; for, the areas of sim. △s are to another as the squares for the corresponding sides.

In sim. △s, when any side of one △ is double that of the other, the area of one △ = 4 times that of the other.

20. 331. Sim. Polygons may be divided into the same no. of sim △s, having the same R. to one another that the polygons have; and the polygons have to one another the duplicate R. of that which their homol. sides have.

COR. 1. Sim figures of four, or of any no. of sides. are to one another in the duplicate R. of their homol. sides.

COR. 2. If three Ls. be propls., then, the first shall be to the 3rd *as* any pol. on the 1st *to* the sim. and similarly described pol. on the 2nd.

26. 346. If two sim. parallelograms have a com. ∠, and be similarly situated, they are about the same diam.

SCH. This Prop. the converse of 24. VI.

27. 347. Of all ▱s applied to the same st. line, and *deficient* by ▱s sim. and similarly situated to that which is described upon the half of the L.; that which is applied to the half, and is sim. to the defect, is the greatest.

Or, Of all the rectangles contd. by the segs. of a given st. L., the greatest is the square described on half the L.

USE. In a given △ to inscribe the greatest parallelogram possible, having an ∠ in common with the △.

31. 363. *Important*. In rt. ∠d △s, the rectil. fig. described on the side opp. to the rt. ∠, is = the sim. and similarly described figures on the sides containing the right ∠.

SCH. 1. A very comprehensive Prop., but one still more general is,—If any ▱s be described on the two sides of any △, and if the sides of the ▱s be produced to meet, and if that . of intersection and the vertex of the △ be joined, and the L. produced; then these ▱s are eq. in area to a ▱ described on the base, and having two of its sides parallel to the L. produced through the . of intersection and the vertex, and limited by the sides of the two ▱s.

2. A circle with the hyp. of a rt. ∠d △ for diam. is eqnal in area to the two ⊙s, having the other two sides for diameters.

32. 366. If two △s which have two sides of the one propl. to two sides of the other, be joined at one ∠, so as to have their homol. sides par. to one another; the rem. sides shall be in a st. L.

SCH. The position of the given sides, not homol., must form an ∠ at the . of junction.

33. 367. In eq. ⊙s, ∠s, whether at the centres, or ⊙ces, have the same R. which the ⊙ces, on which they stand, have to one another; so also have the sectors.

COR. 1. The sectors are to each other as their ∠s.

2. Sim. sectors of the same or eq. ⊙s are equal.

3. An ∠ at the centre of a ⊙ is to four rt ∠s *as* the arc on which it stands to the ⊙ce of the ⊙.

4. In different ⊙s the arcs of eq. ∠s at the centres or ⊙ces are similar.

5. Hence, sim. segments are contained by sim. arcs, and *vice versâ*.

SCH. Some Editors of Euclid substitute the three following Propositions, applicable *to the same* or *equal circles*.

SUBSIDIARY PROPOSITIONS.

USE. To solve the Prob.;—Given four st. Ls, any three of which are together > the fourth, to construct a quadril., of which the sides shall be = those four given st. Ls., in a given order, each to each, and of which also its angular .s lie in the ⊙ce of a ⊙.

F. 378. If a seg. of a ⊙ be bisected, and from the extremities of the base of the seg., and from the . of bisection, st Ls be drawn to any . in the ⊙ce; the sum of the two Ls from the extremities of the base, will have to the L. from the . of bisection the same R. which the base of the seg. has to the base of half the seg.

G. 379. If two . s be taken in the diam. of a ⊙, or of the diam. produced, such that the rect. contained by the segs. intercepted between them and the cen. be eq. to the sq. of the semidiam.; and if from these . s two st. Ls be inflected to any . whatever in the ⊙ce of the ⊙, then the R. of the Ls inflected will be the same with the R. of the segs. intercepted between the two first-mentioned . s and the ⊙ce of the ⊙.

COR. 1. In the fig. ∠ FBE is bisected by AB.
2. Also the ext. vert. ∠ EBG is bisected by BC.

H. 381. If from one extr. of the diam. of a ⊙ a chord be drawn, and a perp. cut both the diam. and the chord, either intr. or extr., the rect. under the diam. and its seg. reckoned from the extremity is eq. to the rect. under the chord and its corresponding seg.

K. 381. If the ∠s at the base of a △ be bisected by two Ls that meet, and the ext. ∠s at the base, formed by producing the two sides, be sim. bisected; then the two . s of concourse, and the vertex shall be in one st. L. which shall bis. the vert. ∠.

L. 383. In a △, as with last Pr., K, the segs. of each side produced that are intercepted between the vertex and the extn. perps. are each = the semiperimeter of the △; the segs. of these sides next the rect. are = the excess of the semiperimeter above the base; and the seg. of each of these sides next the base is respectively = the excess of the semiperimeter above the other side.

REMARKS.

CORRIGENDA.

The errors to be found in Mathematical works, even in those of high repute prove the difficulty of avoiding them, especially when signs and abbreviations are freely used. Besides, the printing was undertaken by those who were not accustomed to mathematical work; and there was also an unavoidable change of workmen; hence without much real blame to any one, the mistakes are more than they would have been; few, however, extending to any error in the reasoning.

Along with the remark in the Preface on the same subject,—will the Reader accept the quaint apology of an old Editor of Euclid? "Some mistakes will still remain, which when thou chancest to meet with correct with the pen, so shalt thou do right to the Author, and supply the defects of the Revisor, and in both doe good to thyself."—RUDD'S *Euclides Elements.*

N.B.—In Column 1, the page is given, P; in Col. 2, the line, L, the arabic numeral on the *left* hand denoting the line from the *top* of the page,—that on the *right*, the line from the *bottom*; in Col. 3, the correction, corr.; and in Col, 4, the printed error, err.

P.	L.	corr.	*for* err.
8	20	Pst. 2.	3
11	8	16, I.	10, I.
17	2	FC;	FG;
39	19	AC–CB;	=CB;
44	1	∴	∵
82	4	A	AD
85	1	FD	ED
99	12	DEB	DEF
104	17	*rad.*	*diam.*
108	6	in L;	in I;
110	2	▭;	□;
115	19	∠CAD	∠CAG
	22	DB	CB
120	6	∴	∵
126	4	arc E &c.	F&c.
128	6	30, III.	28, III.
172	15	KC	KH
176	20	∴ B	∴
178	7	contains	=
		GH	GA
179	11	C=F	C=E
182	11	or $<$	or $>$
187	14	$>$	$<$
188	1	: C,	: A.
	12	F,	E,
192	7	C : D,	B : E,
194	3	D $>$	B $>$
197	20	$\frac{2}{3}$	$\frac{2}{4}$
198	2	$> n$D	$< n$D
201	10	KX,	KH,
202	14	C:D	C:C

P.	L.	corr.	*for* err.
205	3	or $<$	or $>$
206	8	DF;	DE;
207	13	CF	CE
208	6	A′,	A,
209	12	*ay*	*by*
210	8	DF (*bis*)	EF
211	7	$\frac{a+b}{b}$	$\frac{a+b}{d}$
214	8	& 4th,	& 5th,
219	5	22	32
221	8	$\frac{a'}{c'}$	$\frac{a'}{b'}$
222	14	A 12	H 12
230	10	is compd.	and
235	7	continued	contained
236	11	$\neq$B	$\neq$B′
	11	B′;	B;
	10	B	B′
	8	A′	A
238	5	B	R
	4	of B	of R
250	8	DE,	EF,
255	1	are as	are
261	14	∵	∴
264	21	GD,	CD,
272	3,4	CB	EB
	6	C	E
289	2	HC,	AC,
302	5,11	BC;	AC;
	12	ADC,	ABC,

P.	L.	corr.	*for* err.
311	2	triangles	angles
320	6	Cor. 3.	Cor. 33.
325	14	∵	∴
329	11	*omit* EF or	
330	3	6480	3240
331	1	6480	3240
	6	.969	.973
335	12	AB	CB
	15	AC	AB
336	11	$\frac{2\times2}{1}$	$\frac{2\times2}{2}$
348	12	∴	∵
353	18	∵	∴
355	4	EB	EL
	8	36, I.	31, I.
356	9	*To exscribe*	
357	21	10, I.	31, I.
358	8	46, I.	45, I.
362	6	EC	EB
	15	CE	AE
364	12	∵	∴
365	9	33, I.	31, I.
375	7	5, IV.	4, VI.
393	11	equil.	equal
402	13	4 times	4 lines
406	1	5 and	and
413	1	93	97
421	8	=A;	=;
423	10	C : D	B : E
428	2	up into	up
432	8	47, 48,	47, 47.

ERRATA,

Involving an Error of Reasoning.

P.	L.	
46	1 & 3	remove the bracket.
208	16	B ∾ C, C ∾ D, *for* ∾ B : C CD
255	1	*add* ∴ $A^2 : A.B = A.B : B^2$.
259	14	BG : AG *for* AG : BG
270	17 & 20	GB *for* CB
276	7	A*c*, A*d* for *bc*, *cd*

P.	L.	
332	3	*read* in the duplicate ratio of AB : FG.
354	6	The full argument not given.
359	16	(*read*) times with a remainder.
430	7	(*read*) times with a remainder.

FINIS.